IOP Publishing New J. Phys. 26 (2024) 093036 https://doi.org/10.1088/1367-2630/ad748e

New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics
Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

RECEIVED
6 June 2024

REVISED
31 July 2024

ACCEPTED FOR PUBLICATION
28 August 2024

PUBLISHED
25 September 2024

PAPER

Field theory of active Brownian particles with dry friction

Ziluo Zhang[1,2], Shurui Yuan[1,3] and Shigeyuki Komura[1,4,5,*]

1 Wenzhou Institute, University of Chinese Academy of Sciences, Wenzhou, Zhejiang 325001, People's Republic of China
2 Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, People's Republic of China
3 Harbin Institute of Technology, Harbin 150001, People's Republic of China
4 Oujiang Laboratory, Wenzhou, Zhejiang 325000, People's Republic of China
5 Department of Chemistry, Graduate School of Science, Tokyo Metropolitan University, Tokyo 192-0397, Japan
* Author to whom any correspondence should be addressed.

E-mail: komura@wiucas.ac.cn

Keywords: active matter, active Brownian particle, dry friction, field theory, diffusion coefficient, Fokker–Planck equation

Abstract

We present a field theoretic approach to capture the motion of a particle with dry friction for one- and two-dimensional (2D) diffusive particles, and further expand the framework for 2D active Brownian particles. Starting with the Fokker–Planck equation and introducing the Hermite polynomials as the corresponding eigen-functions, we obtain the actions and propagators. Using a perturbation expansion, we calculate the effective diffusion coefficient in the presence of both wet and dry frictions in a perturbative way via the Green–Kubo relation. We further compare the analytical result with the numerical simulation. Our result can be used to estimate the values of dry friction coefficient in experiments.

1. Introduction

Active matter [1–4] refers to particles or creatures that can absorb energy from the environment and change their mechanical state by using the energy. The collective behaviors of animals such as swarming [5, 6], flocks [7–11], and bacteria [12] can be described effectively by appropriate models. Furthermore, artificial active colloidal particles [13, 14] are created experimentally to explore active systems. There are plenty of analytical works from the Stokes–Einstein–Sutherland relation of Brownian motion [15] to active systems [16, 17]. However, fitting analytical results with experimental ones is always a challenge. For example, the restricted confinement can significantly affect the diffusion coefficient [18]. Furthermore, the motion of particles can be influenced by dry friction (Coulomb friction) with the substrate if the size and weight of the particle are large enough. In this case, the buoyancy and thermal fluctuations can barely push the particle away from the substrate. Unlike the wet friction that is always proportional to the velocity, dry friction depends only on the velocity direction.

In the most of the previous studies, dry and wet frictions are not considered together. The particles experiencing dry friction are mostly unaffected by the thermal noise, since the particles are large enough. Moreover, the gravity acting on the particles is usually smaller than thermal noise, and the particles hardly reach the lower surface. Hence, dry friction of such particles can also be neglected. For mesoscopic particles, however, both wet and dry frictions can not be neglected because the gravity drags the particles and their random motions take place at the bottom surface [19, 20]. A one-dimensional (1D) passive particle with dry friction has been discussed [21–23], and further two-dimensional (2D) problem has been also addressed [24].

In this work, we are motivated to explore the motion of an active particle with dry friction. In particular, we obtain the effective diffusion coefficient of a 2D active Brownian particle (ABP) with both dry and wet frictions by using the Doi-Peliti field theory [25–27]. A 2D ABP moves with a finite velocity, and its director undergoes rotational diffusion [28–30]. We first introduce the Langevin equations to describe the motion of an ABP, and later convert it to the corresponding Fokker–Planck equation. Since the effective diffusion coefficient can be calculated via the velocity–velocity correlation function by using the Green–Kubo relation [15], we only consider the particle velocity in this work.

In section 2, we introduce the mathematical basis of this work, such as the equation of motion and its eigen-system to simplify the calculation. In section 3, we use the field-theoretic framework to capture the motion of a 1D diffusing particle with dry friction in a perturbative way. We expand our framework to 2D space in section 4 by approximating dry friction to be isotropic. By treating the self-propulsion of an ABP as a perturbative part, we obtain the corresponding effective diffusion coefficient in section 5. We also compare the 2D result with the simulation result. A summary of our work and some discussion are given in section 6.

2. Model

In this section, we firstly show the mathematical basis of our work. Our main aim is to calculate the effective diffusion coefficient D_{eff} as a function of dry friction coefficient μ. The diffusion coefficient D_{eff} can be extracted from the mean squared displacement in the long-time limit. In this work, we obtain the effective diffusion coefficient by the Green–Kubo relation [15],

$$D_{\mathrm{eff}} = \lim_{t\to\infty} \frac{\left\langle \left(\mathbf{x}(t) - \mathbf{x}(0)\right)^2 \right\rangle}{2dt} = \frac{1}{d}\int_0^\infty \mathrm{d}t \, \langle \mathbf{v}(t)\cdot\mathbf{v}(0)\rangle \ , \tag{1}$$

where the $\langle \bullet \rangle$ is the ensemble average and d is the space dimension.

We describe the motion of an ABP with dry friction by the Langevin equations

$$m\dot{\mathbf{v}}(t) = -\Gamma\mathbf{v}(t) - F\frac{\mathbf{v}(t)}{||\mathbf{v}(t)||} + \Gamma\mathbf{w}_\theta(t) + \boldsymbol{\Xi}(t) \quad \text{with} \quad \left\langle \boldsymbol{\Xi}(t)\boldsymbol{\Xi}^{\mathrm{T}}(t') \right\rangle = 2D\Gamma^2 \mathbb{1}_2 \delta(t-t') \quad \text{and} \quad \langle \boldsymbol{\Xi} \rangle = \mathbf{0}\,, \tag{2a}$$

$$\dot{\theta}(t) = \zeta(t) \quad \text{with} \quad \langle \zeta(t)\zeta(t')\rangle = 2D_{\mathrm{r}}\delta(t-t') \quad \text{and} \quad \langle\zeta(t)\rangle = 0\,, \tag{2b}$$

$$\dot{\mathbf{x}} = \mathbf{v}\,, \tag{2c}$$

where $||\mathbf{v}||$ is the norm of the vector $\mathbf{v}$, $\mathbf{w}_\theta(t) = w[\cos\theta, \sin\theta]^{\mathrm{T}}$ is the self-propelled velocity of the particle which stays on a ring of a radius of w, and m is the mass of the particle. Furthermore, $\Gamma\mathbf{v}(t)$ is the viscous force proportional to the velocity and $F\mathbf{v}(t)/||\mathbf{v}(t)||$ is dry friction force whose magnitude is a constant. And $\mathbb{1}_2$ is a 2D unit matrix. Each component of $\boldsymbol{\Xi}$ and also ζ are Gaussian white noise [31] with zero mean and variance $D\Gamma^2$ and D_{r}, respectively, where D and D_{r} are the diffusion and rotational diffusion coefficients respectively. And each component of $\boldsymbol{\Xi}$ has zero correlation with ζ. Finally, $\mathbf{x}$ is the position of the particle, and we further define the dry friction term $\mathbf{v}/||\mathbf{v}|| = \lim_{\boldsymbol{\epsilon}\to\mathbf{0}} \mathbf{v}/||\mathbf{v}+\boldsymbol{\epsilon}||$ to avoid the zero denominator.

Introducing the rescaled friction coefficients by the mass $\gamma = \Gamma/m$ and $\mu = F/m$, we modify the Langevin equation as

$$\dot{\mathbf{v}}(t) = -\gamma\mathbf{v}(t) - \mu\frac{\mathbf{v}(t)}{||\mathbf{v}(t)||} + \gamma\mathbf{w}_\theta(t) + \boldsymbol{\xi}(t) \quad \text{with} \quad \left\langle \boldsymbol{\xi}(t)\boldsymbol{\xi}^{\mathrm{T}}(t') \right\rangle = 2D\gamma^2 \mathbb{1}_2 \delta(t-t') \quad \text{and} \quad \langle \boldsymbol{\xi}(t) \rangle = \mathbf{0}\,, \tag{3a}$$

$$\dot{\theta}(t) = \zeta(t) \quad \text{with} \quad \langle \zeta(t)\zeta(t')\rangle = 2D_{\mathrm{r}}\delta(t-t') \quad \text{and} \quad \langle\zeta(t)\rangle = 0\,, \tag{3b}$$

whose corresponding Fokker–Planck equation is [32, 33]

$$\partial_t P(\mathbf{v},\theta,t) = \mathcal{L}P(\mathbf{v},\theta,t) \quad \text{with} \quad \mathcal{L} = D\gamma^2\nabla_{\mathbf{v}}^2 + \gamma\nabla_{\mathbf{v}}\cdot\mathbf{v} + \mu\nabla_{\mathbf{v}}\cdot\frac{\mathbf{v}}{||\mathbf{v}||} - \mathbf{w}_\theta\cdot\nabla_{\mathbf{v}} + D_{\mathrm{r}}\partial_\theta^2\,. \tag{4}$$

Since the diffusion coefficient can be extracted from the velocity–velocity correlation function, we drop the positional variable $\mathbf{x}$ in the Fokker–Planck equation, and we only consider the particle in the velocity space. The effective diffusion coefficient of an isolated 2D ABP in free space is well known [34, 35], in our case, it presents the zero dry friction limit $\mu = 0$. We write it below for the later comparison

$$D_0 = D + \frac{w^2}{2D_{\mathrm{r}}}\,. \tag{5}$$

The following functions are used later to simplify the calculation,

$$u_n(v) = \mathrm{e}^{-\frac{v^2}{2\Omega^2}} He_n\left(\frac{v}{\Omega}\right)\,, \tag{6a}$$

$$\tilde{u}_n(v) = \frac{1}{\sqrt{2\pi}n!} He_n\left(\frac{v}{\Omega}\right)\,, \tag{6b}$$

where $He_n(x)$ is the n-th order of the *probabilist's Hermite polynomials* and $\Omega^2 = D\gamma$ [36, 37]. Our definition of $He_n(x)$ corresponds to $2^{-\frac{n}{2}}\texttt{HermiteH}[n, x/\sqrt{2}]$ [38].

The orthogonality relation between $u_n(v)$ and $\tilde{u}(v)$ is

$$\int \mathrm{d}v u_n(v)\,\tilde{u}_m(v) = \Omega\delta_{n,m}\ , \tag{7}$$

where $\delta_{n,m}$ is the Kronecker delta, and $u_n(v)$ are the eigenfunctions of the operator

$$\Omega\partial_v^2 u_n(v) + \partial_v[v u_n(v)] = -n u_n(v)\ . \tag{8}$$

The velocity–velocity correlation function with the initial velocity $\mathbf{v}_0$, direction θ_0 at time t_0 can be calculated by

$$\langle \mathbf{v}(t)\cdot\mathbf{v}(t')\rangle = \langle \mathbf{v}\cdot\mathbf{v}'\rangle = \int \mathrm{d}^2v\,\mathrm{d}^2v' \int_0^{2\pi} \mathrm{d}\theta\,\mathrm{d}\theta'\ \mathbf{v}\mathcal{G}(\mathbf{v},\theta,t|\mathbf{v}',\theta',t')\,\mathbf{v}'\mathcal{G}(\mathbf{v}',\theta',t'|\mathbf{v}_0,\theta_0,t_0)\ , \tag{9}$$

where we use $\mathbf{v}(t) = \mathbf{v}$ and $\mathbf{v}(t') = \mathbf{v}'$ to simplify the notations and $\mathcal{G}(\mathbf{v},\theta,t|\mathbf{v}',\theta',t')$ is the probability density of finding a particle at velocity $\mathbf{v}$ with the self-propulsion direction θ at time t, when the given initial state is at velocity $\mathbf{v}'$ with the propulsion director θ' at time t'.

Since we choose the stationary state as the initial state, the most right propagator $\mathcal{G}(\mathbf{v}',\theta',t'|\mathbf{v}_0,\theta_0,t_0) = \mathcal{G}(\mathbf{v}',\theta')$ does not depend on the initial condition. By using the orthogonality of the Hermite polynomials [39], the above integral can be simplified further, and the result will be presented in the following sections.

3. Diffusive particle in 1D space

We start with the simplest case, i.e. a diffusive particle with dry friction in 1D space. The corresponding Fokker–Planck equation is

$$\partial_t P(v,t) = D\gamma^2\partial_v^2 P(v,t) + \gamma\partial_v v P(v,t) + \mu\partial_v \frac{v}{|v|}P(v,t)\ , \tag{10}$$

where in 1D space, the unit vector $\mathbf{v}/||\mathbf{v}||$ becomes the sign function $v/|v|$ where $|v|$ is the absolute value of v. We further introduce the notations

$$\mathcal{L}_0 = D\gamma^2\partial_v^2 + \gamma\partial_v v\,, \tag{11a}$$

$$\mathcal{L}_\mu = \mu\partial_v\frac{v}{|v|}\ , \tag{11b}$$

similar to the 2D case, we define $v/|v| = \lim_{\epsilon\to 0} v/|v+\epsilon|$ to avoid the zero denominator.

3.1. Action

The corresponding bilinear action and perturbative action are

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_{\text{pert}}\ , \tag{12a}$$

$$\mathcal{A}_0\left(\phi,\tilde{\phi}\right) = \int \mathrm{d}t \int \mathrm{d}v\,\tilde{\phi}(v,t)\left(-\partial_t + \mathcal{L}_0 - r\right)\phi(v,t)\ , \tag{12b}$$

$$\mathcal{A}_{\text{pert}}\left(\phi,\tilde{\phi}\right) = \int \mathrm{d}t \int \mathrm{d}v\,\tilde{\phi}(v,t)\,\mathcal{L}_\mu\phi(v,t)\ . \tag{12c}$$

where r is the death rate to maintain the causality, and will be taken to zero after the inverse Fourier transform. We have introduced the annihilation field ϕ and the Doi-shifted creation field $\tilde{\phi}$ as [25, 27]

$$\phi(v,t) = \frac{1}{\Omega}\int \bar{\mathrm{d}}\omega\ \mathrm{e}^{-\imath\omega t}\sum_{n=0}^{\infty} u_n(v)\,\phi_n(\omega)\ , \tag{13a}$$

$$\tilde{\phi}(v,t) = \frac{1}{\Omega}\int \bar{\mathrm{d}}\omega\ \mathrm{e}^{-\imath\omega t}\sum_{n=0}^{\infty} \tilde{u}_n(v)\,\tilde{\phi}_n(\omega)\ . \tag{13b}$$

Here $\Omega^2 = D\gamma$, $\bar{\mathrm{d}}\omega = \mathrm{d}\omega/(2\pi)$ and further $\bar{\delta}(\omega) = 2\pi\delta(\omega)$, while u and $\tilde{u}$ are consistent with equation (6). Any expectation value can be calculated perturbatively by a path integral [40]

$$\langle\bullet\rangle = \int \mathcal{D}\phi\,\mathcal{D}\tilde{\phi}\bullet\ \mathrm{e}^{\mathcal{A}[\tilde{\phi},\phi]} = \left\langle \bullet\ \mathrm{e}^{\mathcal{A}_{\text{pert}}[\tilde{\phi},\phi]}\right\rangle_0 \quad \text{with} \quad \langle\bullet\rangle_0 = \int \mathcal{D}\phi\,\mathcal{D}\tilde{\phi}\bullet\ \mathrm{e}^{\mathcal{A}_0[\tilde{\phi},\phi]}\,, \tag{14}$$

where we split the action via equation (12*a*). Expanding the exponential with respect to the perturbative part $\mathcal{A}_{\text{pert}}$, we obtain

$$\langle \bullet \rangle = \int \mathcal{D}\phi \mathcal{D}\tilde{\phi}\, \mathrm{e}^{\mathcal{A}_0[\tilde{\phi},\phi]} \bullet \sum_{n=0}^{\infty} \frac{\mathcal{A}_{\text{pert}}^n}{n!} = \left\langle \bullet \sum_{n=0}^{\infty} \frac{\mathcal{A}_{\text{pert}}^n}{n!} \right\rangle_0 . \tag{15}$$

The probability density is the full propagator

$$\mathcal{G}(\mathbf{v},\theta,t|\mathbf{v}_0,\theta_0,t_0) = \left\langle \phi(\mathbf{v},\theta,t)\, \tilde{\phi}(\mathbf{v}_0,\theta_0,t_0) \right\rangle . \tag{16}$$

By plugging the fields equation (13) into the action equation (12*b*), we obtain

$$\mathcal{A}_0 = -\frac{1}{\Omega^2} \int \text{đ}\omega \int \text{đ}\omega' \sum_{n,m=0}^{\infty} \tilde{\phi}_m(\omega')(-\mathring{\imath}\omega + \gamma n + r)\,\phi_n(\omega)\,\text{đ}(\omega+\omega')\,\Omega\delta_{n,m} , \tag{17}$$

where $\Omega\delta_{n,m}$ comes from the orthogonality relation equation (7) between u_n and u_m. The bare propagator can be read off as

$$\left\langle \phi_n(\omega)\tilde{\phi}_m(\omega') \right\rangle_0 = \frac{\Omega\delta_{n,m}\,\text{đ}(\omega+\omega')}{-\mathring{\imath}\omega + \gamma n + r} = \Omega\delta_{n,m}\,\text{đ}(\omega+\omega')G(n,\omega) \triangleq \tag{18}$$

Similarly, we substitute the fields in equation (13) into the perturbative action in equation (12*c*),

$$\mathcal{A}_{\text{pert}} = \sum_{n,m} \frac{\mu}{\Omega^2} \int \text{đ}\omega\text{đ}\omega' \tilde{\phi}_m(\omega')\,\phi_n(\omega) \int \mathrm{d}v \tilde{u}_m(v)\,\partial_v \left[\frac{v}{|v|} u_n(v)\right] \text{đ}(\omega+\omega') \tag{19a}$$

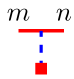
$$= \sum_{n,m} \frac{\mu}{\Omega^2} \int \text{đ}\omega \tilde{\phi}_m(-\omega)\,\phi_n(\omega) \int \mathrm{d}v \left(2\delta(v)\,\tilde{u}_m(v)\,u_n(v) + \frac{v}{|v|}\tilde{u}_m(v)\,\partial_v u_n(v)\right) , \tag{19b}$$

where $\delta(v)$ comes from the derivative of the sign function

$$\partial_v \left(\frac{v}{|v|}\right) = 2\delta(v) . \tag{20}$$

We diagrammatically write the perturbative part of the action as

$$\frac{\mu}{\Omega^2} \int \mathrm{d}v \left(2\delta(v)\tilde{u}_m(v)u_n(v) + \frac{v}{|v|}\tilde{u}_m(v)\partial_v u_n(v)\right) = \Lambda^{m,n} \triangleq \tag{21}$$

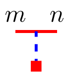


By using the following properties of the Hermite polynomials [36],

$$He_n(v)\,He_m(v) = \sum_{k=0}^{\min(n,m)} \frac{n!m!}{(n-k)!\,(m-k)!k!} He_{n+m-2k}(v) , \tag{22a}$$

$$\partial_v \left(He_n(v)\,\mathrm{e}^{-\frac{v^2}{2}}\right) = -He_{n+1}(v)\,\mathrm{e}^{-\frac{v^2}{2}} , \tag{22b}$$

the analytic form of the above vertex becomes



$$= \frac{\mu}{\Omega^2} \frac{2}{\sqrt{2\pi}m!} \left(He_n(0)\,He_m(0) - \sum_{k=0}^{\min(n+1,m)} \frac{(n+1)!m!}{(n+1-k)!(m-k)!k!} He_{n+m-2k}(0)\right) , \tag{23}$$

where $He_n(0)$ is the 'Hermite zero', which is probabilist's Hermite polynomials evaluated at zero with respect to the n-th order. It is trivial to see that

$$\Lambda^{0,n} = 0 \quad \text{for} \;\; n \in \mathbb{Z}_0^+ , \tag{24}$$

which indicates that a propagator does not have an outgoing index 0 unless it is a bare one. Moreover, since the 'Hermite zero' of arbitrary odd order is zero, $He_{2n+1}(0) = 0$, we find

$$\Lambda^{2m+1,2n} = \Lambda^{2m,2n+1} = 0 \quad \text{for} \;\; n,m \in \mathbb{Z}_0^+ , \tag{25}$$

which shows the outgoing and incoming indices should have the same parity. Otherwise, the corresponding combinations become zero.

In the following, we only consider the perturbative vertices with the limited indices $n, m = 0, 1, 2$, which are

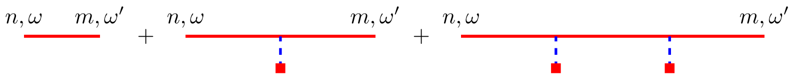
$$\Lambda^{2,0} = \Lambda^{1,1} = \Lambda^{2,2} = -\sqrt{\frac{2}{\pi}}\frac{\mu}{\Omega^2}\,, \tag{26a}$$

$$\Lambda^{0,0} = \Lambda^{0,1} = \Lambda^{1,0} = \Lambda^{0,2} = \Lambda^{1,2} = \Lambda^{2,1} = 0\,. \tag{26b}$$

3.1.1. Propagator

By using the bare propagator and perturbative vertices, the full propagator can be written in a perturbative way as

$$\left\langle \phi_n(\omega)\tilde{\phi}_m(\omega')\right\rangle \triangleq \quad + \quad + \quad + \dots \tag{27a}$$



$$= \bar{\delta}(\omega+\omega')\left\{\Omega\delta_{n,m}G(n,\omega) + \Omega^2 G(n,\omega)\Lambda^{n,m}G(m,\omega) + \dots\right\} \quad . \tag{27b}$$

The full propagators is therefore

$$\left\langle \phi_n(\omega)\,\tilde{\phi}_m(\omega')\right\rangle = \bar{\delta}(\omega+\omega')\sum_{j=0}^{\infty} E_j(n,m,\omega)\ , \tag{28}$$

and the corresponding E_j are given by

$$E_0(n,m,\omega) = \Omega\delta_{n,m}G(n,\omega) \tag{29a}$$

$$E_1(n,m,\omega) = \Omega^2 G(n,\omega)\Lambda^{n,m}G(m,\omega) \tag{29b}$$

$$E_2(n,m,\omega) = \Omega^3 G(n,\omega)\sum_{q=0}^{\infty}\Lambda^{n,q}G(q,\omega)\Lambda^{q,m}G(m,\omega) = \Omega G(n,\omega)\sum_{q=0}^{\infty}\Lambda^{n,q}E_1(q,m,\omega)\ . \tag{29c}$$

We further introduce the recurrence relation between E_j and E_{j+1}

$$E_{j+1}(n,m,\omega) = \Omega G(n,\omega)\sum_{q=0}^{\infty}\Lambda^{n,q}E_j(q,m,\omega) \tag{30}$$

The probability density of velocity v at time t with a given initial state v' at time t' is

$$\mathcal{G}(v,t|v',t') = \frac{1}{\Omega^2}\sum_{n,m=0}^{\infty} u_n(v)\,\tilde{u}_m(v')\lim_{r\downarrow 0}\int \text{đ}\omega\text{đ}\omega'\ \mathrm{e}^{-\hat{\imath}\omega t}\mathrm{e}^{-\hat{\imath}\omega' t'}\left\langle \phi_n(\omega)\,\tilde{\phi}_m(\omega')\right\rangle \tag{31a}$$

$$= \frac{1}{\Omega^2}\sum_{n,m=0}^{\infty} u_n(v)\,\tilde{u}_m(v')\lim_{r\downarrow 0}\int \text{đ}\omega\ \mathrm{e}^{-\hat{\imath}\omega(t-t')}\sum_j E_j(n,m,\omega)\ . \tag{31b}$$

3.1.2. Stationary-state correlation function

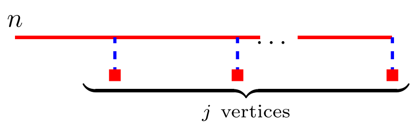
By taking the limit $t_0 \to -\infty$, we obtain the stationary density

$$\mathcal{G}(v) = \lim_{t_0\to-\infty}\left\langle \phi(v,t)\,\tilde{\phi}(v_0,t_0)\right\rangle\ . \tag{32}$$

Previous work shows that taking the stationary limit $t_0 \to -\infty$ and the zero death rate limit $r \downarrow 0$ replaces the incoming index by $\delta_{m,0}$ [35], which physically indicates that the steady-state is independent of the initial condition. Therefore, at stationarity, equation (30) is

$$E_j(n) = \lim_{r\downarrow 0}\lim_{t_0\to-\infty}\int \text{đ}\omega \mathrm{e}^{-\hat{\imath}\omega(t-t_0)}E_j(n,m,\omega) \tag{33}$$

$$= \quad ,$$



where we have dropped the arguments m, ω, since the observable no longer depends on time and initial state. The form of the stationary density is

$$\mathcal{G}(v) = \frac{1}{\Omega^2} \sum_{n=0}^{\infty} u_n(v)\, \tilde{u}_0 \sum_{j=0}^{\infty} E_j(n) \ . \tag{34}$$

Since $\tilde{u}_0(v)$ does not depend on v, we use the notation $\tilde{u}_0 \triangleq \tilde{u}_0(v)$ to indicate the independence of the initial state. We list the first two orders and the recurrence relation of E_j as follows

$$E_0(n) = \Omega \delta_{n,0} \ , \tag{35a}$$

$$E_1(n) = \frac{\Lambda^{n,0}\Omega^2}{n\gamma} \ , \tag{35b}$$

$$E_{j+1}(n) = \frac{\Lambda^{n,q}\Omega}{n\gamma} \sum_{q=1} E_j(q) \ . \tag{35c}$$

Since $\Lambda^{0,n}$ is always zero in equation (24) which vanishes all the propagators with outgoing index $n = 0$ before taking stationary limit, we define $\lim_{n\to 0} \Lambda^{n,m}/n = 0$ for arbitrary m and perform the summation from $q = 1$ to avoid the zero denominators in equation (35).

Our aim is to use the Green–Kubo relation equation (1) to calculate the effective diffusion coefficient. Now, we perform the integral of the velocity–velocity correlation function in the present field theory framework

$$\langle v(t)\, v(t')\rangle = \int \mathrm{d}v \int \mathrm{d}v'\ v \mathcal{G}(v, t|v', t')\, v' \mathcal{G}(v') \ , \tag{36}$$

where we simplify the notation $v = v(t)$ and $v' = v(t')$. By using equations (9), (31), (34) and the orthogonality of the Hermite polynomials [39], the correlation function becomes

$$\langle v(t)\, v(t')\rangle = \int đ\omega\, \mathrm{e}^{-\imath\omega(t-t')} \left\langle \phi_1(\omega)\, \tilde{\phi}_1(-\omega) \right\rangle \left(2\left\langle \phi_2 \tilde{\phi}_0 \right\rangle + \left\langle \phi_0 \tilde{\phi} \right\rangle \right). \tag{37}$$

We first calculate the time-independent observable in the brackets as

$$\left\langle \phi_0 \tilde{\phi}_0 \right\rangle = \Omega \ , \tag{38a}$$

$$\left\langle \phi_2 \tilde{\phi}_0 \right\rangle = \sum_{j=0} E_j(2) = -\Omega \frac{\beta}{2+\beta} + \mathcal{O}\left(\beta^2\right) \ , \tag{38b}$$

where in equation (38b) we only consider the finite number of perturbative vertices up to the second order, and $\mathcal{O}(\beta^2)$ is used to indicate that we do not consider all the contributions higher than the second order of β. The mathematical details are shown in appendix A.1. We introduce a dimensionless parameter

$$\beta = \sqrt{\frac{2}{\pi}} \frac{\mu}{\Omega\gamma} \ . \tag{39}$$

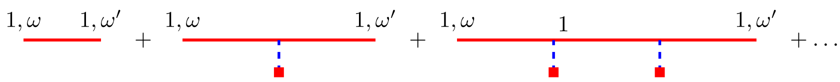
to simplify the notation. Then we have

$$2\left\langle \phi_2(0)\, \tilde{\phi}_0(0) \right\rangle + \left\langle \phi_0(0)\, \tilde{\phi}(0) \right\rangle = \Omega \left(1 - \frac{\beta}{1+\frac{\beta}{2}} \right) + \mathcal{O}\left(\beta^2\right) \ . \tag{40}$$

To calculate the time-dependent observable, we only consider the perturbative part with both incoming and outgoing indices fixed to unity,

$$\int đ\omega \mathrm{e}^{-\imath\omega(t-t')} \left\langle \phi_1(\omega)\tilde{\phi}_1(-\omega) \right\rangle$$

$$= \int đ\omega \mathrm{e}^{-\imath\omega(t-t')}$$



(41a)

$$= \int đ\omega \mathrm{e}^{-\imath\omega(t-t')} \Omega G(1,\omega) \sum_{\ell=0}^{\infty} \left(\Omega G(1,\omega)\Lambda^{1,1}\right)^{\ell} + \mathcal{O}(\beta^2) \tag{41b}$$

$$= \int đ\omega \mathrm{e}^{-\imath\omega(t-t')} \frac{\Omega G(1,\omega)}{1 - \Omega G(1,\omega)\Lambda^{1,1}} + \mathcal{O}(\beta^2) \tag{41c}$$

$$= \int đ\omega\ \mathrm{e}^{-\imath\omega(t-t')} \frac{\Omega}{-\imath\omega + \gamma + \sqrt{\frac{2}{\pi}}\frac{\mu}{\Omega}} + \mathcal{O}(\beta^2) = \Theta(t-t')\Omega \mathrm{e}^{-(\gamma+\gamma\beta)(t-t')} + \mathcal{O}(\beta^2) \ , \tag{41d}$$

where Θ is the Heaviside step function, since the integral converges only for $t \geqslant t'$. Then, the time-correlation function is obtained by substituting equations (40) and (41d) into equation (37),

$$\langle v(t)\, v(t')\rangle = \Theta(t-t')\,\Omega^2 \mathrm{e}^{-\gamma(1+\beta)(t-t')}\left(1-\frac{\beta}{1+\frac{\beta}{2}}\right)+\mathcal{O}\left(\beta^2\right) . \quad (42)$$

By using the Green–Kubo relation in equation (1) and recalling $\Omega^2 = D\gamma$, the effective diffusion coefficient is

$$D_{\mathrm{eff}} = \int_{t'}^{\infty}\mathrm{d}t\ \langle v(t)\, v(t')\rangle = \frac{D}{1+\beta}\left(1-\frac{\beta}{1+\frac{\beta}{2}}\right)+\mathcal{O}\left(\beta^2\right) . \quad (43)$$

If there is no dry friction, $\beta \to 0$, we recover the normal translational diffusion coefficient $D_{\mathrm{eff}}|_{\beta\to 0} = D$, as it should. By expanding the fractions, we obtain the first order correction with respect to the parameter β as

$$D_{\mathrm{eff}} = D(1-2\beta)+\mathcal{O}\left(\beta^2\right) . \quad (44)$$

4. Diffusive particle in 2D space

In 2D space, the corresponding Fokker–Planck equation of a diffusive particle with dry friction is

$$\partial_t P(\mathbf{v},t) = D\gamma^2\nabla_{\mathbf{v}}^2 P(\mathbf{v},t)+\gamma\nabla_{\mathbf{v}}\cdot[\mathbf{v}P(\mathbf{v},t)]+\mu\nabla_{\mathbf{v}}\cdot\left[\frac{\mathbf{v}}{||\mathbf{v}||}P(\mathbf{v},t)\right] . \quad (45)$$

Similar to the 1D case, we split the operator into two parts,

$$\mathcal{L}_0 = D\gamma^2\nabla_{\mathbf{v}}^2+\gamma\nabla_{\mathbf{v}}\cdot\mathbf{v}\,, \quad (46a)$$

$$\mathcal{L}_\mu = \mu\nabla_{\mathbf{v}}\cdot\frac{\mathbf{v}}{||\mathbf{v}||} . \quad (46b)$$

Applying the isotropic property of the first operator $\mathcal{L}_0$, one can simplify the calculation into 1D space with an extra prefactor 2. However, because of the anisotropy of dry friction, there is no such a way to simplify the operator $\mathcal{L}_\mu$. Hence, we use an approximation of the 'sign' function of the velocity $\mathbf{v}$

$$\frac{\mathbf{v}}{||\mathbf{v}||} = \begin{bmatrix}\frac{v_1}{\sqrt{v_1^2+v_2^2}}\\ \\ \frac{v_2}{\sqrt{v_2^2+v_2^2}}\end{bmatrix} = \begin{bmatrix}\frac{v_1}{|v_1|\sqrt{1+\frac{v_2^2}{v_1^2}}}\\ \\ \frac{v_2}{|v_2|\sqrt{1+\frac{v_1^2}{v_2^2}}}\end{bmatrix} \approx \mathrm{e}^{-\frac{1}{4}}\begin{bmatrix}\frac{v_1}{|v_1|}\\ \\ \frac{v_2}{|v_2|}\end{bmatrix} , \quad (47)$$

where we use the following Hermite expansion of the square root in the denominator, the prefactor $\mathrm{e}^{-1/4}$ is the projection of the dry friction term on the He_0, and this prefactor is firstly found approximately from numerical simulation,

$$\frac{1}{\sqrt{1+z^2}} = \mathrm{e}^{-\frac{1}{4}}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k(2k)!}He_{2k}\left(\sqrt{2}\theta\right) , \quad (48)$$

with $\theta = \arctan(z)$. By considering only the zeroth order Hermite polynomial and dropping all the higher order terms, we find the approximation in equation (47). The operator $\mathcal{L}_\mu$ is rewritten as

$$\mathcal{L}_\mu \approx \mathcal{L}'_\mu = \mu\mathrm{e}^{-\frac{1}{4}}\left(\partial_{v_1}\frac{v_1}{|v_1|}+\partial_{v_2}\frac{v_2}{|v_2|}\right) , \quad (49)$$

which is now isotropic. Then, we can simplify this 2D problem to a 1D problem whose corresponding Fokker–Planck equation is

$$\partial_t P^{(2)}(v,t) = D\gamma^2\partial_v^2 P^{(2)}(v,t)+\gamma\partial_v vP^{(2)}(v,t)+\mathrm{e}^{-\frac{1}{4}}\mu\partial_v\frac{v}{|v|}P^{(2)}(v,t) , \quad (50)$$

where we have introduced an upper index $^{(2)}$ to indicate that the above Fokker–Planck equation is obtained from the 2D equation. The corresponding velocity–velocity correlation function is

$$\langle\mathbf{v}(t)\cdot\mathbf{v}(t')\rangle = 2\langle v(t)\, v(t')\rangle , \quad (51)$$

where the prefactor 2 in the RHS is the dimension factor.

The only difference between the Fokker–Planck equation in equation (50) and 1D case in equation (10) is an extra prefactor e−1/4 in the friction term. Therefore, by following the same steps presented in 1D case in section 3, we immediately obtain the velocity–velocity correlation and the effective diffusion coefficient by introducing the modified dimensionless parameter as

$$\tilde{\beta} = \mathrm{e}^{-\frac{1}{4}} \sqrt{\frac{2}{\pi}} \frac{\mu}{\Omega\gamma} , \tag{52}$$

and they are

$$\langle \mathbf{v}(t) \cdot \mathbf{v}(t') \rangle = 2D\gamma \mathrm{e}^{-\gamma(1+\tilde{\beta})(t-t')} \left(1 - \frac{\tilde{\beta}}{1+\frac{\tilde{\beta}}{2}}\right) + \mathcal{O}\left(\tilde{\beta}^2\right) , \tag{53a}$$

$$D_{\mathrm{eff}}^{(2)} = \frac{D}{1+\tilde{\beta}} \left(1 - \frac{\tilde{\beta}}{1+\frac{\tilde{\beta}}{2}}\right) + \mathcal{O}\left(\tilde{\beta}^2\right) . \tag{53b}$$

Here we use an upper index $^{(2)}$ to distinguish the effective diffusion coefficient in 2D from the 1D result in equation (43), where the parameter $\tilde{\beta}$ in the 2D problem is changed from the parameter β in the 1D case.

We compare our result in equation (53*b*) with the numerical simulation. In this work, only finite types of the perturbation vertices are considered, the friction operator is approximated by equation (49), and the result is a perturbation calculation using the first three orders of the Hermite polynomials. We compare the effective diffusion coefficient with the numerical simulation and our field theory approach in figure 1(a), showing a good agreement between them even $\tilde{\beta}$ is large. Additionally, the fourth order correction of the time-independent observable $\left\langle \phi_2 \tilde{\phi}_0 \right\rangle$ is also calculated in appendix A.1.

5. Active Brownian Particles

ABPs are particles that move with constant speed $|\mathbf{w}| = w$ but whose director θ diffuses with the rotational diffusion coefficient D_{r}. The corresponding Fokker–Planck equation of a 2D ABP with dry friction is

$$\partial_t P(\mathbf{v},\theta,t) = \mathcal{L}P(\mathbf{v},\theta,t) \quad \text{with} \quad \mathcal{L} = D\gamma^2 \nabla_{\mathbf{v}} + \gamma \nabla_{\mathbf{v}} \cdot \mathbf{v} + \mu \nabla \cdot \frac{\mathbf{v}}{||\mathbf{v}||} - \gamma \mathbf{w}_\theta \cdot \nabla + D_{\mathrm{r}} \partial_\theta^2 , \tag{54}$$

where $\mathbf{w}_\theta = w[\cos\theta, \sin\theta]^{\mathrm{T}}$.

Similar to the approach presented in section 4, we first use the approximation of the friction term in equations (47) and (49), to approximate the anisotropic operator by an isotropic one. Second, we reduce the problem to 1D and choose the component of $\mathbf{v}$ on the x-axis of the Cartesian plane. The Fokker–Planck equation is then

$$\partial_t P^{(2)}(v,\theta,t) = (\mathcal{L}_0 + \mathcal{L}_\mu + \mathcal{L}_w) P^{(2)}(v,\theta,t) \tag{55}$$

with

$$\mathcal{L}_0 = D\gamma^2 \partial_v^2 + \gamma \partial_v v + D_{\mathrm{r}} \partial_\theta^2 , \tag{56a}$$

$$\mathcal{L}_\mu \approx \mathcal{L}'_\mu = \mu \mathrm{e}^{-\frac{1}{4}} \partial_v \frac{v}{|v|} , \tag{56b}$$

$$\mathcal{L}_w = -\gamma w \cos\theta \partial_v . \tag{56c}$$

5.1. Action

Accordingly, the bilinear action and perturbative actions are

$$\mathcal{A}_0\left(\phi,\tilde{\phi}\right) = \int \mathrm{d}t \int \mathrm{d}v \int_0^{2\pi} \mathrm{d}\theta \tilde{\phi}(v,\theta,t) (-\partial_t + \mathcal{L}_0 - r) \phi(v,\theta,t) , \tag{57a}$$

$$\mathcal{A}_\mu\left(\phi,\tilde{\phi}\right) = \int \mathrm{d}t \int \mathrm{d}v \int_0^{2\pi} \mathrm{d}\theta \tilde{\phi}(v,\theta,t) \mathcal{L}'_\mu \phi(v,\theta,t) , \tag{57b}$$

$$\mathcal{A}_w\left(\phi,\tilde{\phi}\right) = \int \mathrm{d}t \int \mathrm{d}v \int_0^{2\pi} \mathrm{d}\theta \tilde{\phi}(v,\theta,t) \mathcal{L}_w \phi(v,\theta,t) . \tag{57c}$$

Since there is a new variable θ, the fields in this case are introduced as

$$\phi(v,\theta,t)=\frac{1}{\Omega}\sum_{n=0}^{\infty}\sum_{\alpha=-\infty}^{\infty}\int \text{đ}\omega\,\mathrm{e}^{-\mathring{\imath}\omega t}u_n(v)\,\mathrm{e}^{-\mathring{\imath}\alpha\theta}\phi_{n,\alpha}(\omega)\ , \tag{58a}$$

$$\tilde{\phi}(v,\theta,t)=\frac{1}{2\pi\Omega}\sum_{n=0}^{\infty}\sum_{\alpha=-\infty}^{\infty}\int \text{đ}\omega\,\mathrm{e}^{-\mathring{\imath}\omega t}\tilde{u}_n(v)\,\mathrm{e}^{\mathring{\imath}\alpha\theta}\tilde{\phi}_{n,\alpha}(\omega)\ . \tag{58b}$$

Here, we use Roman letters for the velocity indices and Greek letters for the director indices. Similar to the 1D case, the corresponding bilinear action and the μ-dependent perturbative action are

$$\mathcal{A}_0=-\frac{1}{2\pi\Omega^2}\sum_{n,m=0}^{\infty}\sum_{\alpha,\alpha'=-\infty}^{\infty}\int \text{đ}\omega\text{đ}\omega'\tilde{\phi}_{m,\alpha'}(\omega')\left(-\mathring{\imath}\omega+\gamma n+D_{\mathrm{r}}\alpha^2+r\right)\phi_{n,\alpha}(\omega)\,\delta(\omega+\omega')\,\Omega\delta_{n,m}2\pi\delta_{\alpha,\alpha'}\ , \tag{59a}$$

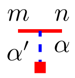
$$\mathcal{A}_\mu=\frac{\mu\mathrm{e}^{-1/4}}{2\pi\Omega^2}\sum_{n,m=0}^{\infty}\sum_{\alpha,\alpha'=-\infty}^{\infty}\int \text{đ}\omega\tilde{\phi}_{m,\alpha'}(-\omega)\,\phi_{n,\alpha}(\omega)\int \mathrm{d}v\left(2\delta(v)\,\tilde{u}_m(v)\,u_n(v)+\frac{v}{|v|}\tilde{u}_m(v)\,\partial_v u_n(v)\right)2\pi\delta_{\alpha,\alpha'}\ , \tag{59b}$$

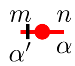
and the self-motility dependent action $\mathcal{A}_w$ is

$$\mathcal{A}_w=-\frac{\gamma w}{2\pi\Omega^2}\sum_{n,m=0}^{\infty}\sum_{\alpha,\alpha'=-\infty}^{\infty}\int \text{đ}\omega\,\tilde{\phi}_{m,\alpha'}(-\omega)\,\phi_{n,\alpha}(\omega)\int \mathrm{d}v\,\tilde{u}_m(v)\,\partial_v u_n(v)\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{e}^{-\mathring{\imath}(\alpha-\alpha')\theta}\cos\theta \tag{60a}$$

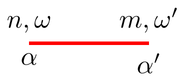
$$=\frac{\gamma w}{2\pi\Omega^2}\sum_{n,m=0}^{\infty}\sum_{\alpha,\alpha'=-\infty}^{\infty}\int \text{đ}\omega\tilde{\phi}_{m,\alpha'}(-\omega)\,\phi_{n,\alpha}(\omega)\,\delta_{n,m-1}2\pi\frac{\delta_{\alpha+1,\alpha'}+\delta_{\alpha-1,\alpha'}}{2}\ . \tag{60b}$$

There are two different types of perturbative vertices. We diagrammatically write the friction-dependent and self-propulsion dependent vertices in the following way

$$\frac{\mu\exp\left(-\frac{1}{4}\right)}{\Omega^2}\int \mathrm{d}v\left(2\delta(v)\tilde{u}_m(v)u_n(v)+\frac{v}{|v|}\tilde{u}_m(v)\partial_v u_n(v)\right)\delta_{\alpha,\alpha'}\triangleq\Lambda^{m,n}_{\alpha',\alpha}\triangleq$$  , (61a)

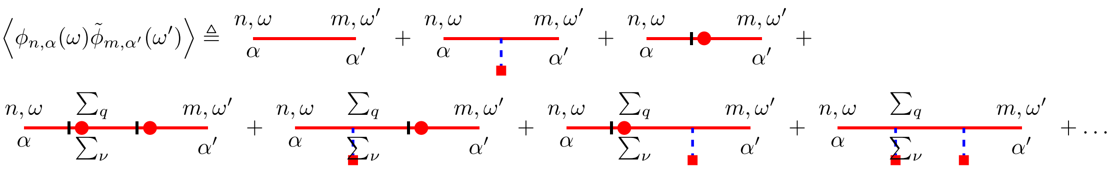
$$\gamma w\frac{\delta_{\alpha+1,\alpha'}+\delta_{\alpha-1,\alpha'}}{2\Omega^2}\delta_{n,m-1}\triangleq\Upsilon^{m,n}_{\alpha',\alpha}\triangleq$$  , (61b)

and the bare propagator is

$$\frac{\Omega\delta_{n,m}\delta_{\alpha,\alpha'}\text{đ}(\omega+\omega')}{-\mathring{\imath}\omega+\gamma n+D_{\mathrm{r}}\alpha^2+r}=\Omega\delta_{n,m}\delta_{\alpha,\alpha'}\text{đ}(\omega+\omega')G(n,\alpha,\omega)\triangleq$$  . (62)

5.2. Propagator

The full propagator is the summation of all the possible combination of the bare propagator in equation (62) and the perturbative vertices in equation (61)

$$\left\langle\phi_{n,\alpha}(\omega)\tilde{\phi}_{m,\alpha'}(\omega')\right\rangle\triangleq$$  (63a)

$$=\text{đ}(\omega+\omega')\Big\{\Omega\delta_{n,m}\delta_{\alpha,\alpha'}G(n,\alpha,\omega)+\Omega^2G(n,\alpha,\omega)(\Upsilon^{\alpha,\alpha'}_{n,m}+\Lambda^{\alpha,\alpha'}_{n,m})G(m,\alpha',\omega)$$
$$+\Omega^3G(n,\alpha,\omega)\sum_{q=0}^{\infty}\sum_{\nu=-\infty}^{\infty}(\Upsilon^{\alpha,\nu}_{n,q}+\Lambda^{\alpha,\nu}_{n,q})G(q,\nu,\omega)(\Upsilon^{\nu,\alpha'}_{q,m}+\Lambda^{\nu,\alpha'}_{q,m})G(m,\alpha',\omega)+\dots\Big\}\ . \tag{63b}$$

Similar to equation (28), we write the full propagator as,

$$\left\langle \phi_{n,\alpha}(\omega)\tilde{\phi}_{m,\alpha'}(\omega')\right\rangle = \bar{\delta}(\omega+\omega')\sum_{j=0}^{\infty}E_j(n,m,\alpha,\alpha',\omega) \ , \tag{64}$$

where the terms shown in equation (63a) are exactly the first three orders of $E_j(n,m,\alpha,\alpha',\omega)$. We list the recurrence relation as

$$E_{j+1}(n,m,\alpha,\alpha',\omega) = \Omega G(n,\alpha,\omega)\sum_{q=0}^{\infty}\sum_{\nu=-\infty}^{\infty}\left(\Upsilon_{n,q}^{\alpha,\nu}+\Lambda_{n,q}^{\alpha,\nu}\right)E_j(q,m,\nu,\alpha',\omega) \ . \tag{65}$$

We therefore obtain the probability density of the velocity at v with director θ at time t with the corresponding initial state (v',θ',t') as

$$\begin{aligned}\mathcal{G}(v,\theta,t|v',\theta',t') &= \frac{1}{2\pi\Omega^2}\sum_{n,m=0}^{\infty}\sum_{\alpha,\alpha'=-\infty}^{\infty}u_n(v)\tilde{u}_m(v')\,\mathrm{e}^{-\imath\alpha\theta}\mathrm{e}^{\imath\alpha'\theta'}\lim_{r\downarrow 0}\int \text{đ}\omega\text{đ}\omega'\bar{\delta}(\omega+\omega') \\ &\quad\times\left\langle \phi_{n,\alpha}(\omega)\tilde{\phi}_{m,\alpha'}(\omega')\right\rangle &(66a)\\ &= \frac{1}{2\pi\Omega^2}\sum_{n,m=0}^{\infty}\sum_{\alpha,\alpha'=-\infty}^{\infty}u_n(v)\tilde{u}_m(v')\,\mathrm{e}^{-\imath\alpha\theta}\mathrm{e}^{\imath\alpha'\theta'}\lim_{r\downarrow 0}\int \text{đ}\omega\mathrm{e}^{-\imath\omega(t-t')} \\ &\quad\times\sum_{j=0}E_j(n,m,\alpha,\alpha',\omega). &(66b)\end{aligned}$$

5.3. Velocity–velocity correlation function

By substituting equation (66b) into equation (9), and using the orthogonality of the Hermite polynomials and the exponential terms, we have

$$\langle v(t)v(t')\rangle = \left\langle \phi_{1,0}(t)\tilde{\phi}_{1,0}(t')\right\rangle\left(2\left\langle\phi_{2,0}\tilde{\phi}_{0,0}\right\rangle+\left\langle\phi_{0,0}\tilde{\phi}_{0,0}\right\rangle\right)+\sum_{\nu=-\infty}^{\infty}\left\langle\phi_{1,0}(t)\tilde{\phi}_{0,\nu}(t')\right\rangle\left\langle\phi_{1,\nu}\tilde{\phi}_{0,0}\right\rangle \ , \tag{67}$$

where the second term comes from the integral over θ'. Since we only consider the first three orders of the speed indices $n,m=0,1,2$ of the perturbation vertices in equation (61), only $\nu=\pm 1$ in the summation are concerned in the second term. We calculate the time-independent observable first by the inverse Fourier transform and take the limits $t_0\to-\infty$ and $r\downarrow 0$

$$\left\langle\phi_{n,\alpha}\tilde{\phi}_{m,\alpha'}\right\rangle = \lim_{r\downarrow 0}\lim_{t_0\to-\infty}\int\text{đ}\omega\mathrm{e}^{-\imath\omega(t-t_0)}\left\langle\phi_{n,\alpha}(\omega)\tilde{\phi}_{m,\alpha'}(-\omega)\right\rangle. \tag{68}$$

We obtain

$$\left\langle\phi_{0,0}\tilde{\phi}_{0,0}\right\rangle = \Omega \ , \tag{69a}$$

$$\left\langle\phi_{2,0}\tilde{\phi}_{0,0}\right\rangle = -\Omega\frac{\tilde{\beta}}{2+\tilde{\beta}}+\frac{w^2\gamma}{4\Omega\left(\gamma+D_\mathrm{r}+\tilde{\beta}\gamma\right)\left(1+\frac{\tilde{\beta}}{2}\right)}+\mathcal{O}\left(\tilde{\beta}^2\right) \ , \tag{69b}$$

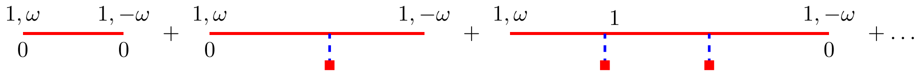
$$\left\langle\phi_{1,1}\tilde{\phi}_{0,0}\right\rangle = \left\langle\phi_{1,-1}\tilde{\phi}_{0,0}\right\rangle = \frac{w\gamma}{2\left(\gamma+D_\mathrm{r}+\tilde{\beta}\gamma\right)}+\mathcal{O}\left(\tilde{\beta}^2\right) \ , \tag{69c}$$

where the details are presented in appendix A.2.

For the time-dependent correlation function, we only consider the lowest perturbation vertices, similar to section 3. The first one is

$$\left\langle \phi_{1,0}(t)\tilde{\phi}_{1,0}(t') \right\rangle = \int đ\omega e^{-\mathring{\imath}\omega(t-t')} \left\langle \phi_{1,0}(\omega)\tilde{\phi}_{1,0}(-\omega) \right\rangle \tag{70a}$$

$$= \int đ\omega e^{-\mathring{\imath}\omega(t-t')}$$



(70b)

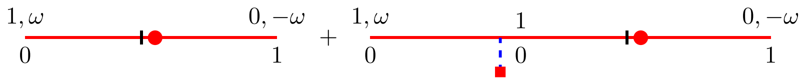
$$= \int đ\omega e^{-\mathring{\imath}\omega(t-t')} \Omega G(1,0,\omega) \sum_{\ell=0}^{\infty} \left(\Omega G(1,0,\omega)\Lambda^{1,1}_{0,0}\right)^{\ell} + \mathcal{O}(\tilde{\beta}^2) \tag{70c}$$

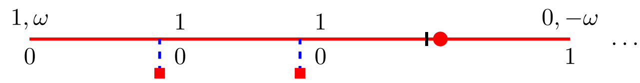
$$= \int đ\omega \; e^{-\mathring{\imath}\omega(t-t')} \frac{\Omega}{-\mathring{\imath}\omega + \gamma + \exp\left(-\frac{1}{4}\right)\sqrt{\frac{2}{\pi}\frac{\mu}{\Omega}}} = \Theta(t-t')\Omega e^{-(\gamma+\gamma\tilde{\beta})(t-t')} + \mathcal{O}(\tilde{\beta}^2) \;, \tag{70d}$$

where $\tilde{\beta}$ is the corresponding dimensionless parameter introduced in equation (52). The second term is

$$\left\langle \phi_{1,0}(t)\tilde{\phi}_{0,1}(t') \right\rangle = \left\langle \phi_{1,0}(t)\tilde{\phi}_{0,-1}(t') \right\rangle = \int đ\omega e^{-\mathring{\imath}\omega(t-t')} \left\langle \phi_{1,0}(\omega)\tilde{\phi}_{0,1}(-\omega) \right\rangle \tag{71a}$$

$$= \int đ\omega e^{-\mathring{\imath}\omega(t-t')}$$





(71b)

$$= \int đ\omega e^{-\mathring{\imath}\omega(t-t')} \Omega^2 G(1,0,\omega)\Upsilon^{1,0}_{0,1} G(0,1,\omega) \sum_{\ell=0}^{\infty} \left(\Omega G(1,0,\omega)\Lambda^{1,1}_{0,0}\right)^{\ell} + \mathcal{O}(\tilde{\beta}^2) \tag{71c}$$

$$= \Theta(t-t') \frac{w\gamma}{2(\gamma+\tilde{\beta}\gamma - D_r)} \left(e^{-D_r(t-t')} - e^{-\gamma(1+\tilde{\beta})(t-t')} \right) + \mathcal{O}(\tilde{\beta}^2) \;. \tag{71d}$$

Then, the velocity–velocity correlation function is

$$\begin{aligned} \langle v(t)\, v(t') \rangle = \Theta(t-t') \Bigg(& \Omega^2 e^{-\gamma(1+\tilde{\beta})(t-t')} \left(1 - \frac{\tilde{\beta}}{1+\frac{\tilde{\beta}}{2}}\right) \\ & + \frac{\gamma w^2 e^{-\gamma(1+\tilde{\beta})(t-t')}}{2\left(\gamma + D_r + \gamma\tilde{\beta}\right)\left(1+\frac{\tilde{\beta}}{2}\right)} + \frac{e^{-D_r(t-t')} - e^{-\gamma(1+\tilde{\beta})(t-t')}}{\gamma + \tilde{\beta}\gamma - D_r} \times \frac{w^2\gamma^2}{2\left(\gamma + D_r + \tilde{\beta}\gamma\right)} \Bigg) + \mathcal{O}\left(\tilde{\beta}^2\right) \;. \end{aligned} \tag{72}$$

By applying the Green–Kubo relation in equation (1), we obtain the effective diffusion coefficient for an ABP with dry friction as

$$D^{(2)}_{\text{eff}} = \frac{D}{1+\tilde{\beta}} \left(1 - \frac{\tilde{\beta}}{1+\frac{\tilde{\beta}}{2}}\right) + \frac{w^2}{2\left(\gamma + D_r + \gamma\tilde{\beta}\right)\left(1+\tilde{\beta}\right)\left(1+\frac{\tilde{\beta}}{2}\right)} + \frac{w^2\gamma}{2D_r\left(\gamma + D_r + \gamma\tilde{\beta}\right)\left(1+\tilde{\beta}\right)} + \mathcal{O}\left(\tilde{\beta}^2\right) \;. \tag{73}$$

When there is no friction, $\tilde{\beta} = 0$, we recover the effective diffusion for an isolated ABP in equation (5) [34, 35, 41, 42],

$$D^{(2)}_{\text{eff}}\big|_{\tilde{\beta}\to 0} = D + \frac{w^2}{2D_r} = D_0 \;. \tag{74}$$

Similar to equation (44), by expanding the fractions, we obtain the first order correction of the effective diffusion coefficient for an ABP with dry friction as

$$D^{(2)}_{\text{eff}} = D\left(1 - 2\tilde{\beta}\right) + \frac{w^2}{2D_r}\left(1 - \frac{3D_r + 4\gamma}{2\left(D_r + \gamma\right)}\tilde{\beta}\right) + \mathcal{O}\left(\tilde{\beta}^2\right) \;. \tag{75}$$

We further compare the analytical result in equation (73) with the numerical simulation in figure 1(b). Even with the existence of the self-propulsion, the agreement between the field theoretic approach and the simulation is still good.

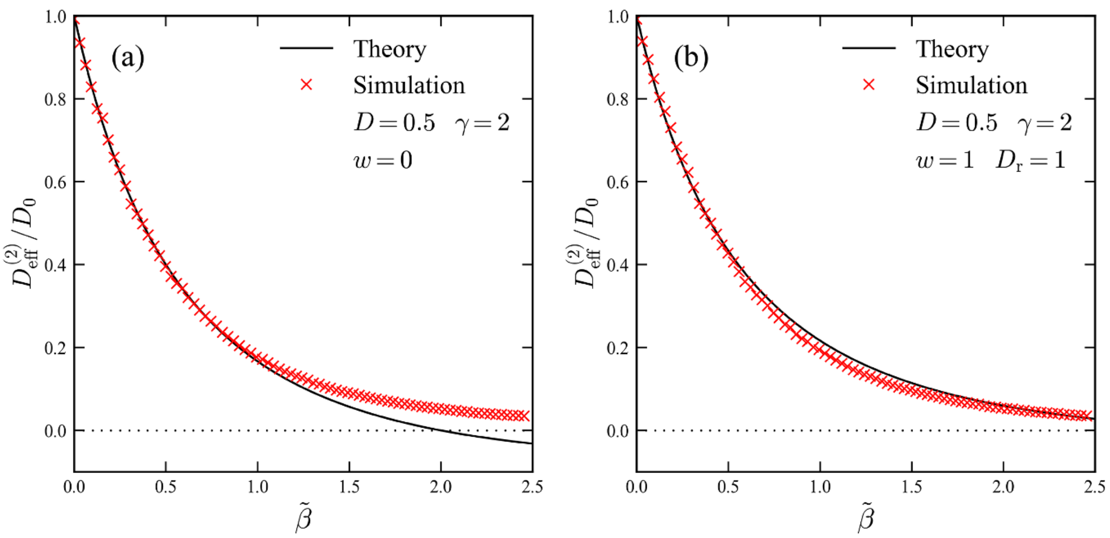


Figure 1. Comparison between the analytical result and numerical simulation. The horizontal axis is the dimensionless parameter $\tilde{\beta}$ in equation (52), and the vertical axis is the ratio between the effective diffusion coefficient $D_{\rm eff}^{(2)}$ in equations (53*b*), (73) and the diffusion coefficient D_0 in the absence of dry friction in equation (74). The numerical result of $D_{\rm eff}^{(2)}$ is obtained from the Langevin equation in equation (3) via the Euler-Maruyama method [43], where the deatils are explained in appendix B. We have used the following parameters: number of the particles $N = 20\,000$, total time $T = 400$, and time step $\Delta t = 0.05$. The initial state of the simulation is $\mathbf{x}(-50) = \mathbf{0}$ and $\theta(-50) = 0$, where we have spent $T_0 = 50$ to wait for the system to reach the stationary state and then started to average the MSD.

6. Summary and discussion

In this paper, we first discussed the 1D diffusing particle with dry friction. From the velocity-corresponding Fokker–Planck equation in equation (10), we determined the bilinear and perturbative actions for a diffusive particle with dry friction in equations (17) and (19), respectively. The full propagator was derived in equation (27). Considering the first three orders of the Hermite polynomials, we only count three types of perturbative vertices with non-zero contributions equation (26). Therefore, the summation in the propagator is reduced to a single term, and the geometric sum is used to obtain the velocity–velocity correlation function in equation (42). Equation (43) was further obtained by using the Green–Kubo relation equation (1).

Then, we have extended this framework to the 2D space. For wet friction, we can always reduce the higher dimensional problem to a 1D problem because the system is isotropic. For dry friction, however, such a treatment is no longer possible because it is anisotropic. We then introduced a Hermite expansion of dry friction, and transformed dry friction into an isotropic operator in equation (47) by considering the leading order. With a prefactor difference, we obtained the 2D effective diffusion coefficient equation (53*b*) from the 1D result in equation (43).

Studying the ABP problem is more complicated since the self-propulsion of the particle provides another type of perturbation vertex. By using the same treatment as in the 1D case, we only considered the limited Hermite order ($n \leqslant 2$). Then, we applied the geometric sums to calculate the velocity–velocity correlation function in equation (72) and further obtained the effective diffusion coefficient in equation (73). Since we have neglected higher orders of the Hermite expansion for both fields and dry friction operator, the diffusion coefficient is a perturbative result. However, the analytical result in equation (73) recovers the numerical simulation very well.

Dry friction force F in the experiment can be estimated by using equation (73) or the first order expansion in equation (75). Substituting back $\Gamma = m\gamma$ and $F = m\mu$, and using the Stokes-Einstein-Sutherland relations $\Gamma = 6\pi\eta a$ and $D = k_{\rm B}T/\Gamma$ [15], where η is the fluid viscosity, a is the radius of the particle, $k_{\rm B}$ is the Boltzmann constant and T is the temperature, we rewrite the dimensionless parameter $\tilde{\beta}$ as

$$\tilde{\beta} = {\rm e}^{-\frac{1}{4}} F\sqrt{\frac{2m}{\pi D\Gamma^3}} = {\rm e}^{-\frac{1}{4}} \frac{F}{6\pi\eta a}\sqrt{\frac{2m}{\pi k_{\rm B}T}}\,. \tag{76}$$

Considering a colloidal particle with a mass $m \approx 10^{-15}$kg and a radius $a \approx 10^{-6}$m in water with the room temperature, and assuming $F \approx 0.1 \times mg$, where g is the gravitational constant, we estimate $\tilde{\beta} \approx 5 \times 10^{-4}$. If the radius is increased to $a \approx 5 \times 10^{-6}$m, the corresponding parameter becomes $\tilde{\beta} \approx 0.14$, which leads to a substantial decrease of the effective diffusion coefficient.

The present work provides a basis for the characterization of the motion of active matter in velocity space via field theory, especially for mesoscopic particles whose diffusion and mass can not be neglected. Because of the wet friction γ, the calculation is simplified by using Hermite expansion of the fields [37] rather than the Fourier transform [35]. This work also provides a basis for ABPs with harmonic interactions. With the aid of the previous work on field theoretic approach of interacting diffusive particles [40, 44], we are currently working on ABPs with non-reciprocal harmonic interactions.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

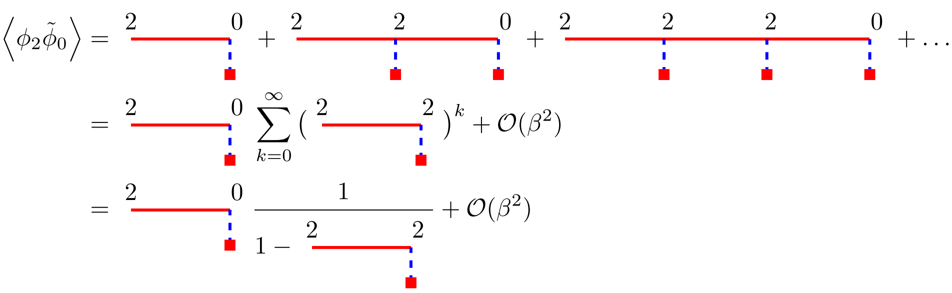
S K acknowledges the support by National Natural Science Foundation of China (Nos. 12274098 and 12250710127) and the startup grant of Wenzhou Institute, University of Chinese Academy of Sciences (No. WIUCASQD2021041). This work was supported by the JSPS Core-to-Core Program 'Advanced core-to-core network for the physics of self-organizing active matter' (JPJSCCA20230002).

Appendix A. Observables

In the following, we show the calculation of the observable $\langle \bullet \rangle$ in detail.

A.1. Purely diffusive case

We first show equation (38). By using equation (24), we immediately obtain equation (38*a*) since the bare propagator is the only contribution. For the observable in the LHS of equation (38*b*), we only consider the limited perturbation vertices listed in equation (26). Diagrammatically, it is



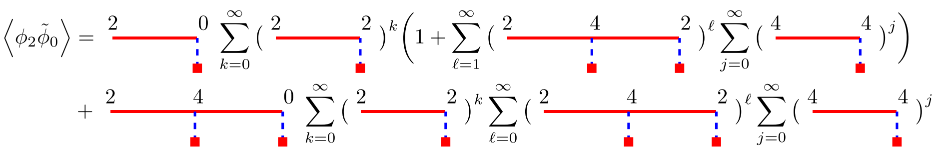
(A1*a*)

(A1*b*)

(A1*c*)

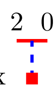
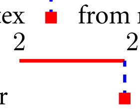
$$= \Omega \frac{\Lambda^{2,0}\Omega}{2\gamma} \frac{1}{1-\frac{\Lambda^{2,2}\Omega}{2\gamma}} + \mathcal{O}(\beta^2) \ . \tag{A1d}$$

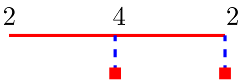
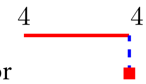
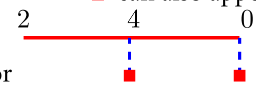
We further include higher order terms up to the fourth order, and the above observable becomes



(A2*a*)

$$= \Omega\frac{\Lambda^{2,0}\Omega}{2\gamma}\frac{1}{1-\frac{\Lambda^{2,2}\Omega}{2\gamma}}\left(1+\frac{\frac{\Lambda^{2,4}\Lambda^{4,2}\Omega^2}{8\gamma^2}}{1-\frac{\Lambda^{2,4}\Lambda^{4,2}\Omega^2}{8\gamma^2}}\frac{1}{1-\frac{\Lambda^{4,4}\Omega}{4\gamma}}\right)+\Omega\frac{\Lambda^{2,4}\Lambda^{4,0}\Omega^2}{8\gamma^2}\frac{1}{1-\frac{\Lambda^{2,2}\Omega}{2\gamma}}\frac{1}{1-\frac{\Lambda^{2,4}\Lambda^{4,2}\Omega^2}{8\gamma^2}}\frac{1}{1-\frac{\Lambda^{4,4}\Omega}{4\gamma}}+\mathcal{O}(\beta^2) \ , \tag{A2b}$$

where the first line of equation (A2a) indicates that the propagator starts with a vertex  from most left, and the first summation shows that there can be arbitrary number of the propagator  connecting to the left of the propagator. The terms in the bracket indicates that, if there is at least one

 connected, the propagator  can also appear as many as possible. Similarly, in the second line of equation (A2a), the propagator  is the leading order, and there can be arbitrary number of the three types of the propagators appearing in the summations. We perform the geometric sum and obtain equation (A2b).

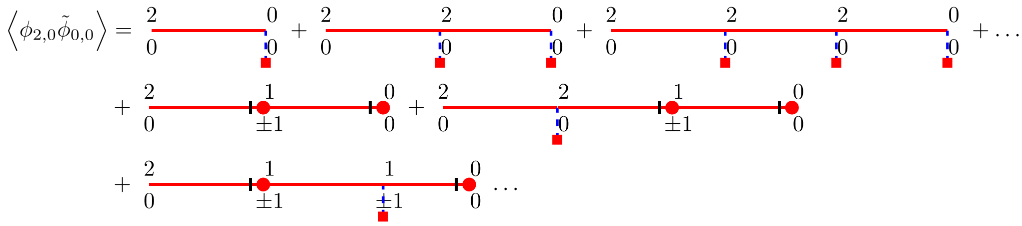
Since the term $\Lambda\Omega/\gamma$ is proportional to the dimensionless parameters β and $\tilde{\beta}$ introduced in equations (39) and (52) for 1D and 2D space, respectively, we obtain the effective diffusion coefficient with the fourth order correction as

$$D_{\text{eff}} = \frac{D}{1+\beta}\left[1 - \frac{\beta}{1+\frac{\beta}{2}}\left(1 - \frac{\frac{\beta^2}{16}}{1+\frac{\beta^2}{16}}\frac{1}{1+\frac{3\beta}{8}}\right) - \frac{\beta^2}{24}\frac{1}{1+\frac{\beta}{2}}\frac{1}{1+\frac{\beta^2}{16}}\frac{1}{1+\frac{3\beta}{8}}\right] + \mathcal{O}\left(\beta^2\right), \quad \text{for 1D} \tag{A3a}$$

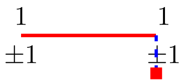
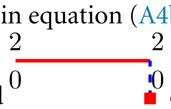
$$D^{(2)}_{\text{eff}} = \frac{D}{1+\tilde{\beta}}\left[1 - \frac{\tilde{\beta}}{1+\frac{\tilde{\beta}}{2}}\left(1 - \frac{\frac{\tilde{\beta}^2}{16}}{1+\frac{\tilde{\beta}^2}{16}}\frac{1}{1+\frac{3\tilde{\beta}}{8}}\right) - \frac{\tilde{\beta}^2}{24}\frac{1}{1+\frac{\tilde{\beta}}{2}}\frac{1}{1+\frac{\tilde{\beta}^2}{16}}\frac{1}{1+\frac{3\tilde{\beta}}{8}}\right] + \mathcal{O}\left(\tilde{\beta}^2\right), \quad \text{for 2D.} \tag{A3b}$$

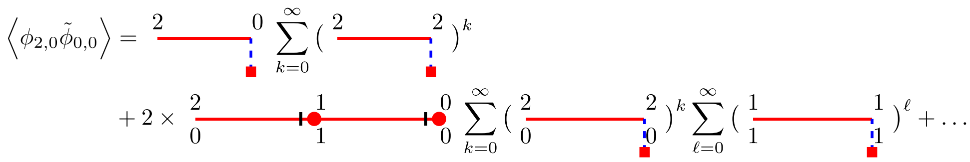
A.2. ABP case

In the following, we obtain equation (69). Similar to the pure diffusive case, $\left\langle \phi_{0,0}\tilde{\phi}_{0,0}\right\rangle$ in equation (69*a*) is a bare propagator without any perturbative vertices. We write the LHS of equation (69*b*) diagrammatically



(A4*a*)
(A4*b*)
(A4*c*)

where the first line is exactly equation (A1), which is a combination of the bare propagator and the friction dependent perturbation only. The second and third lines includes the self-propulsion perturbation part, where the first term in equation (A4b) is the leading order. The rest terms show that arbitrary number of  and  can be inserted into the diagram in a suitable position. Therefore, we write equation (A4)

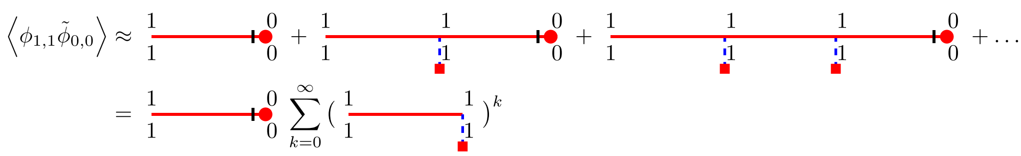


(A5*a*)

$$= \Omega\frac{\Lambda^{2,0}\Omega}{2\gamma}\frac{1}{1-\frac{\Lambda^{2,2}\Omega}{2\gamma}} + 2\Omega\frac{\Upsilon^{2,1}_{0,1}\Upsilon^{1,0}_{1,0}\Omega^2}{2\gamma(\gamma+D_{\text{r}})}\frac{1}{1-\frac{\Lambda^{2,2}_{0,0}\Omega}{2\gamma}}\frac{1}{1-\frac{\Lambda^{1,1}_{1,1}\Omega}{\gamma+D_{\text{r}}}} + \mathcal{O}(\tilde{\beta}^2), \tag{A5b}$$

where the prefactor 2 is a symmetry factor that comes from ± 1.

Similary, for the observable in equation (69*c*), we write it by the diagrams as



(A6*a*)
(A6*b*)

$$= \Omega\frac{\Upsilon^{0,0}_{1,0}\Omega}{\gamma+D_{\text{r}}}\frac{1}{1-\frac{\Lambda^{1,1}_{1,1}\Omega}{\gamma+D_{\text{r}}}} + \mathcal{O}(\tilde{\beta}^2), \tag{A6c}$$

By plugging equation (61) and the dimensionless parameter equation (52) into equations (A5*a*) and (A6*a*) , we obtain equations (69*b*) and (69*c*), respectively.

Appendix B. Numerical simulation

In this paper, we use the Euler-Maruyama method to simulate the system via the corresponding Langevin equation equation (3),

$$\mathbf{v}_{i+1}-\mathbf{v}_i=-\gamma\mathbf{v}_i\Delta t-\mu\frac{\mathbf{v}_i}{||\mathbf{v}_i||}\Delta t+\gamma\mathbf{w}_{\theta_i}\Delta t+\sqrt{2D\gamma^2\Delta t}\boldsymbol{\xi}_i\,, \tag{B1a}$$

$$\theta_{i+1}-\theta_i=\sqrt{2D_\mathrm{r}\Delta t}\zeta_i\,, \tag{B1b}$$

$$\mathbf{x}_{i+1}-\mathbf{x}_i=\mathbf{v}_i\Delta t\,, \tag{B1c}$$

with a suitable initial condition. In the above, $\boldsymbol{\xi}_i$ is a 2D vector and the components are Gaussian random variables with zero expected value and unit variance. Similarly, ζ_i are also Gaussian random variables with zero expected value and unit variance. The effective diffusion coefficient is obtained from the mean-squared displacement

$$D^{(2)}_\mathrm{eff}=\frac{1}{4NM\Delta t}\sum_{i=1}^{N}\left(\mathbf{x}^{(i)}_M-\mathbf{x}^{(i)}_0\right)^2\,, \tag{B2}$$

where M is the total steps and $M\Delta t=T$ is the total time, and N is the number of the particles to obtain the ensemble average.

ORCID iDs

Ziluo Zhang https://orcid.org/0000-0002-7477-8208
Shigeyuki Komura https://orcid.org/0000-0003-3422-5745

References

[1] Cates M E 2012 Diffusive transport without detailed balance in motile bacteria: does microbiology need statistical physics? *Rep. Prog. Phys.* **75** 042601
[2] Bechinger C, Di Leonardo R, Löwen H, Reichhardt C, Volpe G and Volpe G 2016 Active particles in complex and crowded environments *Rev. Mod. Phys.* **88** 045006
[3] Marchetti M C, Joanny J F, Ramaswamy S, Liverpool T B, Prost J, Rao M and Aditi Simha R 2013 Hydrodynamics of soft active matter *Rev. Mod. Phys.* **85** 1143–89
[4] Ramaswamy S 2010 The mechanics and statistics of active matter *Annu. Rev. Condens. Matter Phys.* **1** 323–45
[5] Mach R and Schweitzer F 2007 Modeling vortex swarming in daphnia *Bull. Math. Biol.* **69** 539–62
[6] Vollmer J, Vegh A, Lange C and Eckhardt B 2006 Vortex formation by active agents as a model for daphnia swarming *Phys. Rev.* E **73** 061924
[7] Cavagna A, Giardina I and Grigera T S 2018 The physics of flocking: Correlation as a compass from experiments to theory *Phys. Rep.* **728** 1–62
[8] Reynolds C 1987 Flocks, herds and schools: a distributed behavioral model *ACM SIGGRAPH Comput. Graph.* **21** 25–34
[9] Toner J and Tu Y 1995 Long-range order in a two-dimensional dynamical XY model: How birds fly together *Phys. Rev. Lett.* **75** 4326–9
[10] Toner J and Tu Y 1998 Flocks, herds and schools: A quantitative theory of flocking *Phys. Rev.* E **58** 4828–58
[11] Vicsek T, Czirók A, Ben-Jacob E, Cohen I and Shochet O 1995 Novel type of phase transition in a system of self-driven particles *Phys. Rev. Lett.* **75** 1226–9
[12] Berg H C 2004 *E. Coli in Motion* (Springer)
[13] Yaouen F and Cristina Marchetti. M 2012 Athermal phase separation of self-propelled particles with no alignment *Phys. Rev. Lett.* **108** 235702
[14] Huang Y, Zhang Z, Yang L and He Q 2024 Typhoon eye-shaped global convective flow field-induced colloidal motor swarm *Colloids Surfaces* A **689** 133692
[15] Doi M 2013 *Soft Matter Physics* (OUP Oxford)
[16] Tailleur J and Cates M E 2008 Statistical mechanics of interacting run-and-tumble bacteria *Phys. Rev. Lett.* **100** 218103
[17] Roberts C and Pruessner G 2022 Exact solution of a boundary tumbling particle system in one dimension *Phys. Rev. Res.* **4** 033234
[18] Avni Y, Komura S and Andelman D 2021 Brownian motion of a charged colloid in restricted confinement *Phys. Rev.* E **103** 042607
[19] Hui X, Mengmeng S, Xinjian F, Zhihua L, Weinan C, Lei W, Lixin D and Qiang H 2019 Reconfigurable magnetic microrobot swarm: multimode transformation, locomotion and manipulation *Sci. Robot.* **4** eaav8006
[20] Lin Z, Tieyan S, Zhiguang W, Gao C, Lin X and Qiang H 2017 Light-activated active colloid ribbons *Angew. Chemie Int., Edn.* **56** 13517–20
[21] de Gennes P G 2005 Brownian motion with dry friction *J. Stat. Phys.* **119** 953–62
[22] Hayakawa H 2005 Langevin equation with coulomb friction *Physica* D **205** 48–56
[23] Touchette H, Van der Straeten E and Just W 2010 Brownian motion with dry friction: Fokker-planck approach *J. Phys. A: Math. Theor.* **43** 445002
[24] Das P, Puri S and Schwartz M 2017 Single particle brownian motion with solid friction *Eur. Phys. J.* E **40** 60
[25] Doi M 1976 Second quantization representation for classical many-particle system *J. Phys. A: Math. Gen.* **9** 1465–77
[26] Peliti L 1985 Path integral approach to birth-death processes on a lattice *J. Phys.-Paris* **46** 1469–83

[27] Cardy J 2008 Reaction-Diffusion Processes *Non-Equilibrium Statistical Mechanics and Turbulence* (Cambridge University Press) pp 108–61

[28] Cates M E and Tailleur J 2013 When are active Brownian particles and run-and-tumble particles equivalent? Consequences for motility-induced phase separation *Europhys. Lett.* **101** 20010

[29] Kurzthaler C, Leitmann S and Franosch T 2016 Intermediate scattering function of an anisotropic active brownian particle *Sci. Rep.* **6** 36702

[30] Stenhammar J, Marenduzzo D, Allen R J and Cates M E 2014 Phase behaviour of active Brownian particles: the role of dimensionality *Soft Matter* **10** 1489–99

[31] van Kampen N G 2007 *Stochastic Processes in Physics and Chemistry* 3rd edn (Elsevier)

[32] Risken H 1989 *The Fokker-Planck equation: Methods of Solutions and Applications* 2nd edn (Springer)

[33] Pavliotis G A 2014 *Stochastic Processes and Applications Diffusion Processes, the Fokker-Planck and Langevin Equations* (Springer)

[34] Cates M and Tailleur J 2015 Motility-induced phase separation *Annu. Rev. Condens. Matter Phys.* **6** 219–44

[35] Zhang Z, Fehértói-Nagy L, Polackova M and Pruessner G 2024 Field theory of active brownian particles in potentials *New J. Phys.* **26** 013040

[36] Abramowitz M and Stegun I A (eds) 1970 *Handbook of Mathematical Functions* (Dover Publications, Inc.)

[37] Garcia-Millan R and Pruessner G 2021 Run-and-tumble motion in a harmonic potential: field theory and entropy production *J. Stat. Mech.* 063203

[38] Wolfram Research Inc. 2014 *Mathematica* (Wolfram Research, Inc.) Version 10.0.2.0

[39] Garcia-Millan R 2020 Interactions, correlations and collective behaviour in nonequilibrium systems *PhD Thesis* Imperial College London

[40] Pruessner G and Garcia-Millan R 2022 Field theories of active particle systems and their entropy production (arXiv:2211.11906)

[41] Scholz C, Jahanshahi S, Ldov A and Löwen H 2018 Inertial delay of self-propelled particles *Nat. Commun.* **9** 5156

[42] Patel M and Chaudhuri D 2023 Exact moments and re-entrant transitions in the inertial dynamics of active brownian particles *New J. Phys.* **25** 123048

[43] Kloeden P E and Platen E 2011 *Numerical Solution of Stochastic Differential Equations* (Springer)

[44] Zhang Z and Garcia-Millan R 2023 Entropy production of nonreciprocal interactions *Phys. Rev. Res.* **5** L022033