

Time-correlation functions for odd Langevin systems

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ABSTRACT

We investigate the statistical properties of fluctuations in active systems that are governed by nonsymmetric responses. Both an underdamped Langevin system with an odd resistance tensor and an overdamped Langevin system with an odd elastic tensor are studied. For a system in thermal equilibrium, the time-correlation functions should satisfy time-reversal symmetry and the antisymmetric parts of the correlation functions should vanish. For the odd Langevin systems, however, we find that the antisymmetric parts of the time-correlation functions can exist and that they are proportional to either the odd resistance coefficient or the odd elastic constant. This means that the time-reversal invariance of the correlation functions is broken due to the presence of odd responses in active systems. Using the short-time asymptotic expressions of the time-correlation functions, one can estimate an odd elastic constant of an active material such as an enzyme or a motor protein.

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I. INTRODUCTION

Over the last decades, various active systems such as motor proteins, bacteria, flocks of birds, and fishes were intensively studied as fundamental problems of nonequilibrium statistical mechanics and biophysics.¹ Recently, investigations have begun to characterize these active systems with nonsymmetric response functions such as odd viscosity or odd elasticity.^{2–5} In thermal equilibrium, response functions such as resistance coefficient and elastic modulus need to satisfy certain symmetry properties. For example, the resistance coefficient tensor of a rigid object in a viscous fluid should be a symmetric matrix owing to time-reversal symmetry of low-Reynolds-number hydrodynamic fluid.^{6,7} Such a special property is one example of more general Onsager's reciprocal relations that restrict the symmetry of transport coefficient matrices within the linear response theory. For nonequilibrium active systems, however,

such reciprocal relations are often violated, and the response functions generally consist of both symmetric (even) and antisymmetric (odd) parts.

One of the nonreciprocal responses that have been investigated is the odd viscosity proposed by Avron some years ago.⁸ In general, viscosity is a fourth-rank tensor that linearly connects a stress tensor and a rate-of-strain tensor. According to Onsager's reciprocal theorem, the viscosity tensor should be symmetric for ordinary passive fluids under the exchange of pairs of indices. For an active suspension of rotary motor, however, such symmetry is violated and an odd part of the viscosity can exist.^{2,9} The microscopic origin of odd viscosity is attributed to the broken time-reversal symmetry of the constituent elements. Recently, several people derived the generalized Green–Kubo relation for odd viscosity that arises when the time-reversal symmetry of stress fluctuation is violated.^{10–12} Some of the present authors calculated the resistance

tensor of a two-dimensional (2D) liquid domain immersed in a fluid with odd viscosity and showed that it has nonzero antisymmetric components.^{13,14}

Recently, Scheibner *et al.* introduced the concept of odd elasticity to describe non-conserved interactions in active materials.³ An elastic modulus is, in general, a fourth-rank tensor that linearly connects stress and strain tensors. For a passive system, it should be symmetric under the exchange of pairs of indices because elastic forces are generally conservative.¹⁵ For active systems, on the other hand, an elastic modulus can have both even and odd parts.^{3,16} Importantly, the active moduli quantify the amount of work extracted along quasistatic strain cycles. In our recent paper, we used the variational principle of the Onsager–Machlup integral to describe the stochastic dynamics of a micromachine with odd elasticity.¹⁷ Furthermore, the concept of odd viscoelasticity¹⁸ and odd diffusion tensor has also been reported.¹⁹

In this paper, we investigate the statistical properties of fluctuations in active systems that are governed by nonsymmetric responses. We employ linear Langevin equations with odd responses and obtain various time-correlation functions.^{20–24} Generally, symmetry properties of time-correlation functions can be discussed in terms of their transformation under time-translational and time-reversal operations.^{6,7} When the system is in a steady state, the correlation functions need to satisfy the time-translational invariance. Although the time-reversal invariance of the cross-correlation functions is also satisfied in a thermal equilibrium situation,⁷ we show that such symmetry is violated in the presence of nonsymmetric responses. Two limiting cases will be investigated in detail; linear Langevin systems with a nonsymmetric resistance tensor and a nonsymmetric elastic tensor. In both cases, the time-reversal symmetry of correlation functions is violated in the presence of odd responses. We show that one can estimate an odd elastic constant of an active material such as an enzyme or a motor protein by using the short-time asymptotic expressions of the time-correlation functions. As an example of this application, we demonstrate the estimation of odd elasticity from the numerical simulation of our enzyme model with chemical reactions.²⁵

In Sec. II, we briefly summarize the symmetry properties of time-correlation functions both in equilibrium and out-of-equilibrium. In Sec. III, we investigate the linear Langevin system with an odd resistance tensor. In Sec. IV, we discuss the Langevin system with an odd elastic tensor. In Sec. V, we explain the application of our results to the model enzyme system. A summary and further discussions are provided in Sec. VI.

II. TIME-CORRELATION FUNCTIONS

Let us introduce N -dimensional position variables $x_\alpha(t)$ ($\alpha = 1, 2, \dots, N$) and velocity variables $v_\alpha(t) = \dot{x}_\alpha(t)$, where the dot indicates the time derivative. The variables $x_\alpha(t)$ represent, for example, positions of colloid particles in a suspension or structural parameters of a protein molecule. Considering only the fluctuations, we assume that the averages of $x_\alpha(t)$ and $v_\alpha(t)$ vanish, i.e., $\langle x_\alpha(t) \rangle = 0$ and $\langle v_\alpha(t) \rangle = 0$, where $\langle \dots \rangle$ indicates the ensemble average. Note that $x_\alpha(t)$ is even and $v_\alpha(t)$ is odd under time-reversal transformation.

Let us define the following position–position, position–velocity, and velocity–velocity time-correlation matrices:

$$\phi_{\alpha\beta}(t) = \langle x_\alpha(t)x_\beta(0) \rangle, \quad (1)$$

$$\chi_{\alpha\beta}(t) = \langle v_\alpha(t)x_\beta(0) \rangle, \quad (2)$$

$$\psi_{\alpha\beta}(t) = \langle v_\alpha(t)v_\beta(0) \rangle. \quad (3)$$

The equal-time-correlation functions for $t = 0$ are defined with a bar such as $\bar{\phi}_{\alpha\beta} = \phi_{\alpha\beta}(t = 0)$ and we similarly define $\bar{\chi}_{\alpha\beta}$ and $\bar{\psi}_{\alpha\beta}$.

Generally, one can decompose the time-correlation matrices into symmetric and antisymmetric parts as

$$\phi_{\alpha\beta}(t) = \phi_{\alpha\beta}^S(t) + \phi_{\alpha\beta}^A(t), \quad (4)$$

$$\chi_{\alpha\beta}(t) = \chi_{\alpha\beta}^S(t) + \chi_{\alpha\beta}^A(t), \quad (5)$$

$$\psi_{\alpha\beta}(t) = \psi_{\alpha\beta}^S(t) + \psi_{\alpha\beta}^A(t), \quad (6)$$

where, for example, $\phi_{\alpha\beta}^S(t) = \phi_{\beta\alpha}^S(t)$ and $\phi_{\alpha\beta}^A(t) = -\phi_{\beta\alpha}^A(t)$ hold. Note that the mathematical meanings of the superscripts “S” and “A” are the same as “even” and “odd,” respectively. However, we employ “S” and “A” for time-correlation functions, whereas “even” and “odd” are used for viscosity and elasticity due to convention.⁵ In the following, we argue the properties of the above time-correlation functions when time-translational invariance and time-reversal invariance are satisfied.

A. Time-translational invariance

If the system is in a steady state (both in equilibrium and out-of-equilibrium), time-correlation functions do not depend on the origin of time and satisfy the relation $\langle a(t)b(t') \rangle = \langle a(t-t')b(0) \rangle$.^{6,7} As a result of such time-translational invariance, we have

$$\langle a(t)b(0) \rangle = \langle a(0)b(-t) \rangle = \langle b(-t)a(0) \rangle. \quad (7)$$

Hence, the position–position and velocity–velocity correlation matrices satisfy the following relations:

$$\phi_{\alpha\beta}(t) = \phi_{\beta\alpha}(-t), \quad (8)$$

$$\psi_{\alpha\beta}(t) = \psi_{\beta\alpha}(-t). \quad (9)$$

Concerning the position–velocity correlation function, we have $\chi_{\alpha\beta}(t) = \dot{\phi}_{\alpha\beta}(t)$ (and also $\psi_{\alpha\beta}(t) = -\dot{\chi}_{\alpha\beta}(t)$). Hence, time-translational invariance requires that

$$\chi_{\alpha\beta}(t) = -\chi_{\beta\alpha}(-t). \quad (10)$$

The above symmetry properties in the steady state can be conveniently expressed in terms of the symmetric and antisymmetric parts of the correlation functions as

$$\phi_{\alpha\beta}^S(t) = \phi_{\alpha\beta}^S(-t), \quad \phi_{\alpha\beta}^A(t) = -\phi_{\alpha\beta}^A(-t), \quad (11)$$

$$\chi_{\alpha\beta}^S(t) = -\chi_{\alpha\beta}^S(-t), \quad \chi_{\alpha\beta}^A(t) = \chi_{\alpha\beta}^A(-t), \quad (12)$$

$$\psi_{\alpha\beta}^S(t) = \psi_{\alpha\beta}^S(-t), \quad \psi_{\alpha\beta}^A(t) = -\psi_{\alpha\beta}^A(-t). \quad (13)$$

In other words, $\phi_{\alpha\beta}^S(t)$ and $\psi_{\alpha\beta}^S(t)$ are even functions of time, while $\phi_{\alpha\beta}^A(t)$ and $\psi_{\alpha\beta}^A(t)$ are odd functions.⁶ On the other hand, $\chi_{\alpha\beta}^S(t)$ and $\chi_{\alpha\beta}^A(t)$ are odd and even functions of time, respectively.

For the equal-time-correlation functions, we set $t = 0$ in Eqs. (8)–(10),

$$\bar{\phi}_{\alpha\beta} = \bar{\phi}_{\beta\alpha}, \quad \bar{\chi}_{\alpha\beta} = -\bar{\chi}_{\beta\alpha}, \quad \bar{\psi}_{\alpha\beta} = \bar{\psi}_{\beta\alpha}. \quad (14)$$

Hence, $\bar{\phi}_{\alpha\beta}$ and $\bar{\psi}_{\alpha\beta}$ are symmetric matrices with respect to an exchange of indices, while $\bar{\chi}_{\alpha\beta}$ is a completely antisymmetric matrix.

B. Time-reversal invariance in thermal equilibrium

Next, we discuss the properties of the correlation functions in thermal equilibrium when time-reversal invariance holds.^{6,7} In this situation, the correlation functions satisfy the relation $\langle a(t)b(0) \rangle_{\text{eq}} = \varepsilon_a \varepsilon_b \langle a(-t)b(0) \rangle_{\text{eq}}$, where $\varepsilon_{a(b)}$ takes the value 1 or -1 depending on the time-reversal symmetry of the variable $a(b)$. For example, we have $\varepsilon_x = 1$ and $\varepsilon_v = -1$, as mentioned before. Therefore, the equilibrium correlation functions need to satisfy the following symmetry relations:

$$\phi_{\alpha\beta}^{\text{eq}}(t) = \phi_{\alpha\beta}^{\text{eq}}(-t), \quad (15)$$

$$\chi_{\alpha\beta}^{\text{eq}}(t) = -\chi_{\alpha\beta}^{\text{eq}}(-t), \quad (16)$$

$$\psi_{\alpha\beta}^{\text{eq}}(t) = \psi_{\alpha\beta}^{\text{eq}}(-t). \quad (17)$$

In thermal equilibrium, time-translational invariance is also satisfied and hence Eqs. (11)–(13) hold simultaneously. Then, the antisymmetric parts of the time-correlation matrices should vanish in equilibrium,

$$\phi_{\alpha\beta}^{A,\text{eq}}(t) = \chi_{\alpha\beta}^{A,\text{eq}}(t) = \psi_{\alpha\beta}^{A,\text{eq}}(t) = 0. \quad (18)$$

As a result, the time-correlation matrices have only the symmetric parts that satisfy $\phi_{\alpha\beta}^S(t) = \phi_{\beta\alpha}^S(t)$. Hence, the time-correlation matrices should be symmetric under the exchange of the two indices in thermal equilibrium. In nonequilibrium situations, however, the antisymmetric parts can exist because time-reversal invariance can be violated. Moreover, we also have $\bar{\chi}_{\alpha\beta}^{\text{eq}} = 0$ by considering $t = 0$ in Eq. (16).

III. UNDERDAMPED LANGEVIN SYSTEM WITH ODD RESISTANCE TENSOR

A. N degrees of freedom

In this section, we consider a free Brownian particle embedded in an active chiral fluid that is characterized by odd viscosity, as schematically shown in Fig. 1(a). In a two-dimensional space, the drag force acting on the particle due to the surrounding active fluid is given by a nonsymmetric resistance tensor.^{13,14} To describe the Brownian dynamics of the particle, we use the underdamped Langevin equation in the presence of a nonsymmetric resistance tensor.

The underdamped linear Langevin equation for N velocity variables $v_\alpha(t)$ can be written as^{6,7,20,21}

$$m\dot{v}_\alpha(t) = -\Lambda_{\alpha\beta}v_\beta(t) + mF_{\alpha\beta}\xi_\beta(t), \quad (19)$$

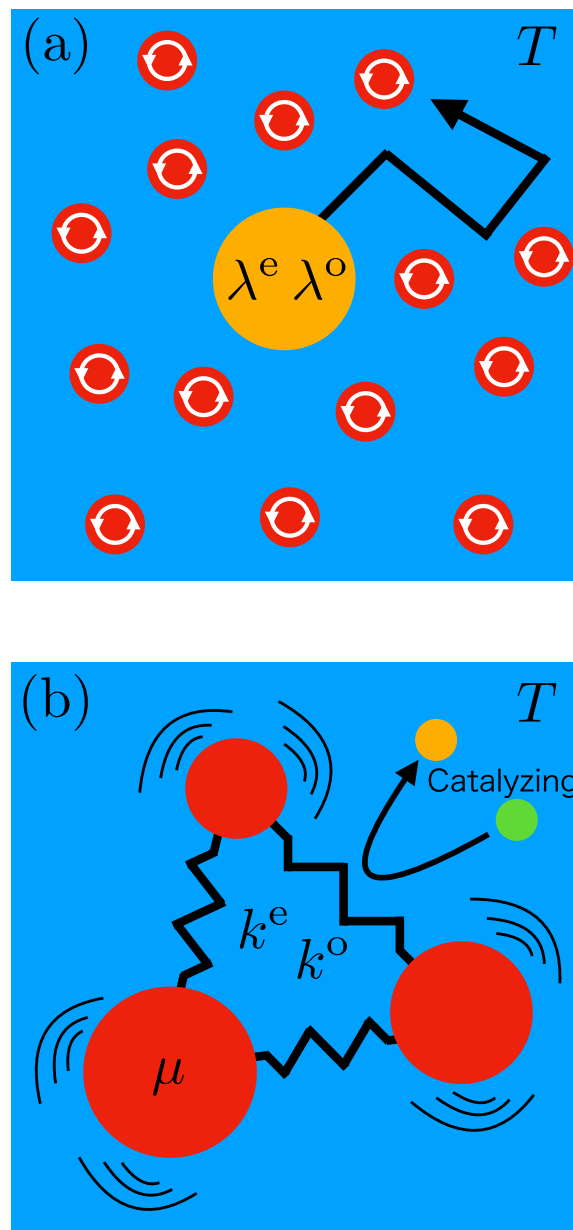


FIG. 1. (a) Diffusion of a Brownian particle (orange circle) that experiences both even and odd resistance forces whose coefficients are given by λ^e and λ^o , respectively. When the surrounding fluid is composed of active chiral elements (red circles), the resistance tensor can have both symmetric and antisymmetric parts [see Eq. (26)]. (b) A coarse-grained model of an enzyme consisting of domains that are connected to springs. A substrate (green circle) changes into a product (orange circle) via a catalytic chemical reaction. Each spring is characterized by the even elastic constant k^e and the odd elastic constant k^o [see Eq. (35)].

where m is the mass of a Brownian particle, $\Lambda_{\alpha\beta}$ is the resistance tensor,^{13,14} $F_{\alpha\beta}$ is the noise amplitude, and $\xi_{\beta}(t)$ is Gaussian white noise that satisfies

$$\langle \xi_{\alpha}(t) \rangle = 0, \quad \langle \xi_{\alpha}(t) \xi_{\beta}(t') \rangle = \delta_{\alpha\beta} \delta(t - t'). \quad (20)$$

The strength of the noise can be conveniently characterized by the symmetric tensor defined by $B_{\alpha\beta} = F_{\alpha\gamma} F_{\beta\gamma} / 2$.

For a passive system, the resistance tensor should be symmetric, $\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha}$, due to Lorentz reciprocal theorem or, more generally, Onsager's reciprocal relations.^{6,7} In addition, the second law of thermodynamics requires that it should be positive definite. For an active system, however, $\Lambda_{\alpha\beta}$ can have an antisymmetric part and we can generally decompose it as^{13,14}

$$\Lambda_{\alpha\beta} = \Lambda_{\alpha\beta}^e + \Lambda_{\alpha\beta}^o. \quad (21)$$

Here, the symmetric (even) part and the antisymmetric (odd) part satisfy $\Lambda_{\alpha\beta}^e = \Lambda_{\beta\alpha}^e$ and $\Lambda_{\alpha\beta}^o = -\Lambda_{\beta\alpha}^o$, respectively. The linear Langevin equation in Eq. (19) can be analytically solved as described in Refs. 20 and 21 and the steps are also briefly summarized in Appendix A.

In thermal equilibrium, the equal-time velocity–velocity correlation function is determined by the equipartition theorem as^{6,7}

$$\overline{\psi}_{\alpha\beta} = \frac{k_B T}{m} \delta_{\alpha\beta}, \quad (22)$$

where k_B is the Boltzmann constant and T the temperature. Then, we solve the Lyapunov equation^{20,21} in Eq. (A5) for the noise strength $B_{\alpha\beta}$ as

$$B_{\alpha\beta} = \frac{k_B T}{m^2} \Lambda_{\alpha\beta}^e. \quad (23)$$

This is the fluctuation dissipation theorem for a passive system in which only $\Lambda_{\alpha\beta}^e$ exists.^{6,7}

Next, we argue the velocity–velocity time-correlation matrix $\psi_{\alpha\beta}(t) = \langle v_{\alpha}(t) v_{\beta}(0) \rangle$. For N degrees of freedom, it is enough to know the short-time behavior of $\psi_{\alpha\beta}(t)$ to discuss its time-reversal symmetry. As derived in Eqs. (A11) and (A12) of Appendix A, we obtain the short-time behavior of $\psi_{\alpha\beta}(t) = \psi_{\alpha\beta}^S(t) + \psi_{\alpha\beta}^A(t)$, where the symmetric and antisymmetric parts become

$$\psi_{\alpha\beta}^S(t) \approx \frac{k_B T}{m} \left(\delta_{\alpha\beta} - \frac{\Lambda_{\alpha\beta}^e |t|}{m} \right), \quad (24)$$

$$\psi_{\alpha\beta}^A(t) \approx -\frac{k_B T}{m^2} \Lambda_{\alpha\beta}^o t. \quad (25)$$

In the above, we have assumed that $|\overline{\Lambda}_i| |t| / m \ll 1$ is satisfied for all the eigenvalues $\overline{\Lambda}_i$ of the matrix $\Lambda_{\alpha\beta}$. In accordance with Eq. (13), $\psi_{\alpha\beta}^S(t)$ is an even function of time and $\psi_{\alpha\beta}^A(t)$ is an odd function. In contrast to Eq. (18), however, $\psi_{\alpha\beta}^A$ does not vanish and time-reversal symmetry is broken when $\Lambda_{\alpha\beta}^o \neq 0$. This is an important consequence when the odd part of the resistance tensor exists.

B. Two degrees of freedom

To perform analytical calculations, we discuss the time-correlation matrix when $N = 2$. We further assume that the resistance tensor is given by the following form:

$$\Lambda_{\alpha\beta} = \lambda^e \delta_{\alpha\beta} + \lambda^o \epsilon_{\alpha\beta}, \quad (26)$$

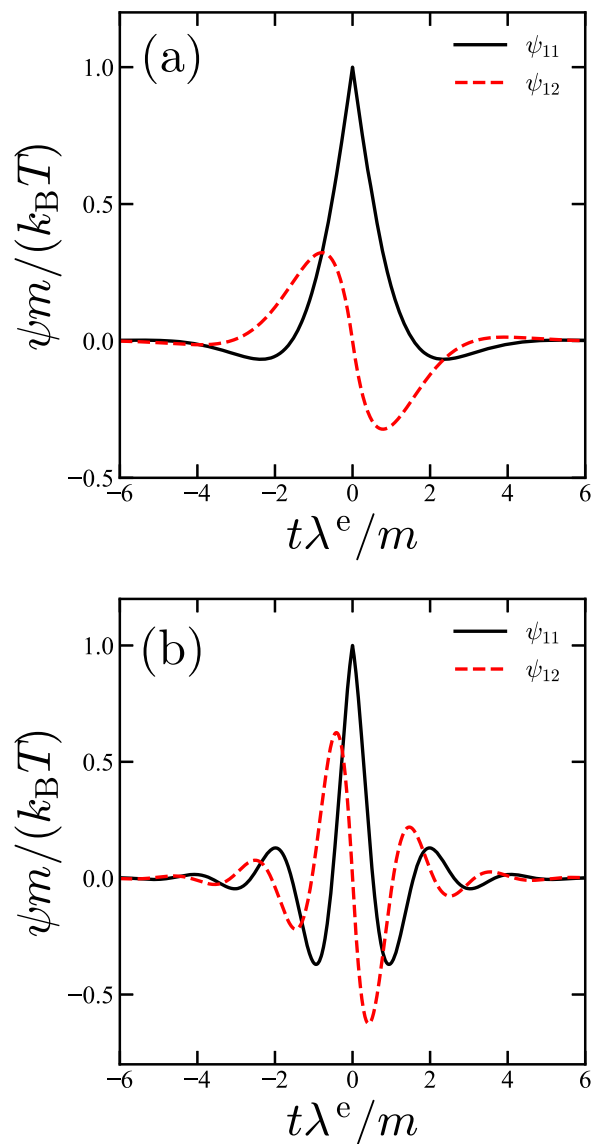


FIG. 2. Plots of the scaled velocity–velocity correlation functions $\psi_{11}(t)$ (black solid line) and $\psi_{12}(t)$ (red dashed line) as a function of dimensionless time $\lambda^e t / m$ when (a) $\lambda^o / \lambda^e = 1$ and (b) $\lambda^o / \lambda^e = 3$ for $N = 2$ [see Eqs. (27) and (28)]. In both cases, $\psi_{11}(t)$ is an even function of time, while $\psi_{12}(t)$ is an odd function of time. Since $\psi_{12}(t)$ violates Eq. (17), time-reversal invariance is broken in the presence of odd resistance tensor. In (b), we observe the oscillatory behavior of the correlation functions. Note that the position–position correlation function $\phi_{\alpha\beta}$ behaves in the same way as $\psi_{\alpha\beta}$ as long as the proper scaling is made.

where $\epsilon_{\alpha\beta}$ is the 2D Levi-Civita tensor with $\epsilon_{11} = \epsilon_{22} = 0$ and $\epsilon_{12} = -\epsilon_{21} = 1$. Note that $\lambda^e > 0$ while λ^o can take both positive and negative values. More general cases are discussed in [Appendix B](#).

The symmetric and antisymmetric parts of $\psi_{\alpha\beta}(t)$ become

$$\psi_{\alpha\beta}^S(t) = \frac{k_B T}{m} e^{-\lambda^e |t|/m} \cos(\lambda^o t/m) \delta_{\alpha\beta}, \quad (27)$$

$$\psi_{\alpha\beta}^A(t) = -\frac{k_B T}{m} e^{-\lambda^e |t|/m} \sin(\lambda^o t/m) \epsilon_{\alpha\beta}. \quad (28)$$

We confirm again that $\psi_{\alpha\beta}^S(t)$ is an even function of time and $\psi_{\alpha\beta}^A(t)$ is an odd function. Also, the existence of $\psi_{\alpha\beta}^A$ when $\lambda^o \neq 0$ indicates the violation of time-reversal symmetry.

In [Fig. 2](#), we plot $\psi_{11}(t)$ (black solid line) and $\psi_{12}(t)$ (red dashed line) as a function of dimensionless time $\lambda^e t/m$. The other parameter is $\lambda^o/\lambda^e = 1$ in [Fig. 2\(a\)](#) and $\lambda^o/\lambda^e = 3$ in [Fig. 2\(b\)](#). In both cases, $\psi_{11}(t)$ is an even function, while $\psi_{12}(t)$ is an odd function. In [Fig. 2\(b\)](#), we observe an oscillatory behavior.

IV. OVERDAMPED LANGEVIN SYSTEM WITH ODD ELASTIC TENSOR

A. N degrees of freedom

As shown in [Fig. 1\(b\)](#), we consider a deformable object such as an enzyme in a passive viscous fluid. We investigate its active dynamics induced by the energy injection as a result of a chemical reaction. We further assume that the odd elastic tensor can describe such a nonequilibrium process as argued in [Refs. 17 and 26](#). To discuss the structural fluctuation of the object driven by thermal motions of the surrounding passive fluid, we employ an overdamped Langevin system with an odd elastic tensor.

The Langevin equation for N position variables $x_\alpha(t)$ can be written as^{6,7,20,21}

$$\dot{x}_\alpha = -M_{\alpha\beta} K_{\beta\gamma} x_\gamma + G_{\alpha\beta} \xi_\beta(t), \quad (29)$$

where $M_{\alpha\beta}$ is the mobility tensor that is symmetric, $M_{\alpha\beta} = M_{\beta\alpha}$, due to Onsager's reciprocal relations for passive fluids. Moreover, $M_{\alpha\beta}$ is positive definite according to the second law of thermodynamics.⁷ In general, the mobility tensor $M_{\alpha\beta}$ is the inverse of the resistance tensor $\Lambda_{\alpha\beta}$ introduced in [Sec. III](#), and $M_{\alpha\beta}$ can also be nonsymmetric for active chiral fluids. However, we do not consider such a general case because we intend to attribute the origin of the nonequilibrium effect to the chemical reaction and not to the activity in the surrounding fluid. As in [Sec. III](#), $\xi_\beta(t)$ is Gaussian white noise that satisfies [Eq. \(20\)](#). The tensor $G_{\alpha\beta}$ represents the noise strength that is further related to the diffusion tensor by $D_{\alpha\beta} = G_{\alpha\gamma} G_{\beta\gamma}/2$ that is symmetric by definition. In our work, we consider stochastic processes driven by thermal fluctuations and assume the relation $D_{\alpha\beta} = k_B T M_{\alpha\beta}$.^{6,7}

In [Eq. \(29\)](#), $K_{\alpha\beta}$ is the elastic constant tensor. For passive systems, $K_{\alpha\beta}$ should be symmetric because elastic forces are conservative.¹⁵ For active systems with non-conservative interactions, however, $K_{\alpha\beta}$ can have an antisymmetric part that corresponds to odd elasticity.^{3,17,26} Hence, $K_{\alpha\beta}$ can generally be written as

$$K_{\alpha\beta} = K_{\alpha\beta}^e + K_{\alpha\beta}^o, \quad (30)$$

where the symmetric (even) part and the antisymmetric (odd) part satisfy $K_{\alpha\beta}^e = K_{\beta\alpha}^e$ and $K_{\alpha\beta}^o = -K_{\beta\alpha}^o$, respectively, similar to the resistance tensor.

For N -dimensional overdamped equations, we obtain the short-time behavior of $\phi_{\alpha\beta}(t) = \phi_{\alpha\beta}^S(t) + \phi_{\alpha\beta}^A(t)$ as [see [Eqs. \(A11\)](#) and [\(A12\)](#) in [Appendix A](#)]

$$\phi_{\alpha\beta}^S(t) \approx \bar{\phi}_{\alpha\beta} - k_B T M_{\alpha\beta} |t|, \quad (31)$$

$$\phi_{\alpha\beta}^A(t) \approx -\frac{1}{2} [M_{\alpha\gamma} K_{\gamma\delta} \bar{\phi}_{\delta\beta} - M_{\beta\gamma} K_{\gamma\delta} \bar{\phi}_{\delta\alpha}] t = \bar{\chi}_{\alpha\beta} t, \quad (32)$$

where $\bar{\chi}_{\alpha\beta} = \langle v_\alpha x_\beta \rangle$ and note the relation $\chi_{\alpha\beta}(t) = \dot{\phi}_{\alpha\beta}(t)$. In the above, we have assumed that $|\overline{MK}_i| |t| \ll 1$ is satisfied for all the eigenvalues \overline{MK}_i of the matrix $M_{\alpha\gamma} K_{\gamma\beta}$. The equal-time-correlation function $\bar{\phi}_{\alpha\beta} = \langle x_\alpha x_\beta \rangle$ obeys the Lyapunov equation,^{20,21}

$$M_{\alpha\gamma} K_{\gamma\delta} \bar{\phi}_{\delta\beta} + M_{\beta\gamma} K_{\gamma\delta} \bar{\phi}_{\delta\alpha} = 2D_{\alpha\beta}. \quad (33)$$

We eliminate $\bar{\phi}_{\alpha\beta}$ from this Lyapunov equation by using $\bar{\chi}_{\alpha\beta}$ in [Eq. \(32\)](#) and obtain

$$M_{\alpha\gamma} K_{\gamma\delta} \bar{\chi}_{\delta\beta} + \bar{\chi}_{\alpha\gamma} K_{\delta\gamma} M_{\delta\beta} = -2k_B T M_{\alpha\gamma} K_{\gamma\delta} M_{\delta\beta}. \quad (34)$$

With this equation, we can prove that $\bar{\chi}_{\alpha\beta} \neq 0$ and, hence, $\phi_{\alpha\beta}^A \neq 0$ when $K_{\alpha\beta}^o \neq 0$. Therefore, the time-reversal symmetry discussed in [Eq. \(17\)](#) is broken due to the presence of $K_{\alpha\beta}^o$.

B. Two degrees of freedom

Alternatively, we discuss the time-correlation functions when $N = 2$. We assume that the elastic tensor is given by the following form:

$$K_{\alpha\beta} = k^e \delta_{\alpha\beta} + k^o \epsilon_{\alpha\beta}. \quad (35)$$

In the following, we further assume that the mobility tensor takes the form $M_{\alpha\beta} = \mu \delta_{\alpha\beta}$. A more general situation is discussed in [Appendix C](#).

The position-position correlation function $\phi_{\alpha\beta}(t) = \langle x_\alpha(t) x_\beta(0) \rangle = \phi_{\alpha\beta}^S(t) + \phi_{\alpha\beta}^A(t)$ is given by

$$\phi_{\alpha\beta}^S(t) = \frac{k_B T}{k^e} e^{-\mu k^e |t|} \cos(\mu k^o t) \delta_{\alpha\beta}, \quad (36)$$

$$\phi_{\alpha\beta}^A(t) = -\frac{k_B T}{k^e} e^{-\mu k^e |t|} \sin(\mu k^o t) \epsilon_{\alpha\beta}. \quad (37)$$

Then, the equal-time-correlation function becomes

$$\bar{\phi}_{\alpha\beta} = \frac{k_B T}{k^e} \delta_{\alpha\beta}, \quad (38)$$

which is independent of k^o . As shown in [Appendix C](#), however, $\bar{\phi}_{\alpha\beta}$ can depend on k^o in a more general situation. The behavior of $\phi_{\alpha\beta}(t)$ is the same as that of $\psi_{\alpha\beta}(t)$ in [Fig. 2](#) as long as the proper scaling is made.

In the short-time limit, i.e., $|\mu k^e t| \ll 1$ and $|\mu k^o t| \ll 1$, Eqs. (36) and (37) become

$$\phi_{\alpha\beta}^S(t) \approx \bar{\phi}_{\alpha\beta} - k_B T \mu |t| \delta_{\alpha\beta}, \quad (39)$$

$$\phi_{\alpha\beta}^A(t) \approx -\frac{k_B T k^o}{k^e} \mu t \epsilon_{\alpha\beta}. \quad (40)$$

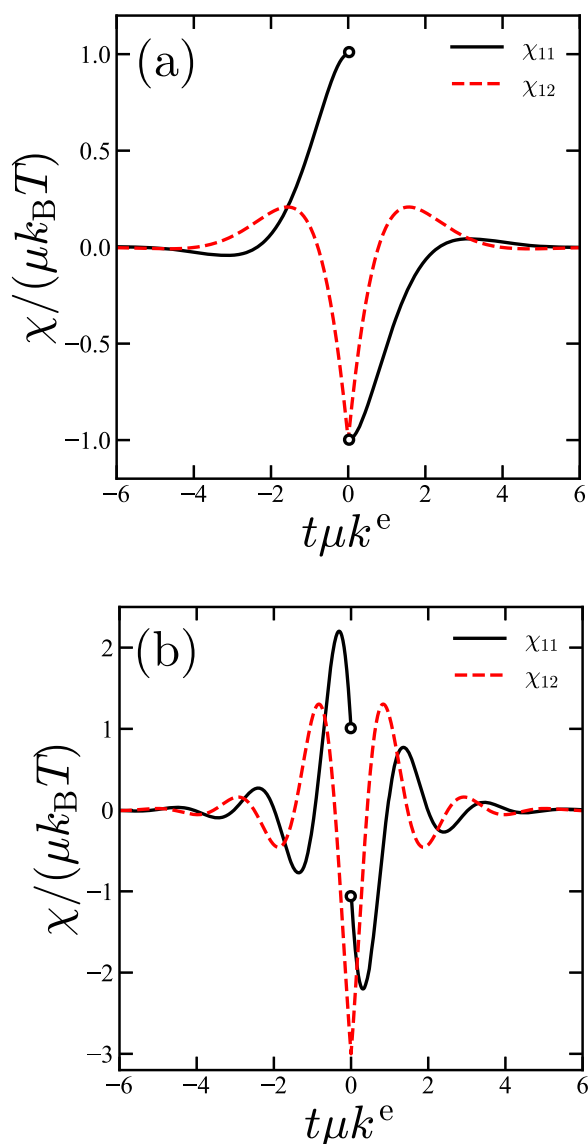


FIG. 3. Plots of the scaled velocity–velocity correlation functions $\chi_{11}(t)$ (black solid line) and $\chi_{12}(t)$ (red dashed line) as a function of dimensionless time $t\mu k^e$ when (a) $k^o/k^e = 1$ and (b) $k^o/k^e = 3$ for $N = 2$ [see Eqs. (41) and (42)]. In both cases, $\chi_{11}(t)$ is an odd function of time and discontinuous at $t = 0$, while $\chi_{12}(t)$ is an even function of time. Since $\chi_{12}(t)$ violates Eq. (16), time-reversal invariance is broken in the presence of odd elastic tensor. In (b), we observe oscillatory behavior of the correlation functions.

The slope of the symmetric part is given by the transport coefficient μ and, hence, it is related to the diffusion coefficient according to the fluctuation dissipation theorem.⁷ On the other hand, the slope of the antisymmetric part is characterized by the ratio k^o/k^e .

Under the same assumptions, we next discuss the position–velocity correlation function $\chi_{\alpha\beta}(t) = \langle v_\alpha(t)x_\beta(0) \rangle = \chi_{\alpha\beta}^S(t) + \chi_{\alpha\beta}^A(t)$, where

$$\chi_{\alpha\beta}^S(t) = -k_B T \mu e^{-\mu k^e |t|} \left[\text{Sgn}(t) \cos(\mu k^o t) + \frac{k^o}{k^e} \sin(\mu k^o t) \right] \delta_{\alpha\beta}, \quad (41)$$

$$\chi_{\alpha\beta}^A(t) = k_B T \mu e^{-\mu k^e |t|} \left[\sin(\mu k^o |t|) - \frac{k^o}{k^e} \cos(\mu k^o t) \right] \epsilon_{\alpha\beta}. \quad (42)$$

In the above, the function $\text{Sgn}(t) = t/|t|$ takes either 1 or -1 depending on its sign. In accordance with Eq. (12), $\chi_{\alpha\beta}^S(t)$ is an odd function of time, while $\chi_{\alpha\beta}^A(t)$ is an even function. On the other hand, the presence of $\chi_{\alpha\beta}^A$ indicates the broken time-reversal symmetry due to odd elasticity.

From Eq. (42), the equal-time-correlation function can be obtained as

$$\bar{\chi}_{\alpha\beta} = -\frac{k_B T k^o}{k^e} \mu \epsilon_{\alpha\beta}, \quad (43)$$

where we have used $\text{Sgn}(0) = 0$. Since $\bar{\chi}_{\alpha\beta}^{\text{eq}} = 0$ should hold when time-reversal invariance is satisfied, the presence of $\bar{\chi}_{\alpha\beta}$ indicates that time-reversal symmetry is broken in the presence of odd elasticity. In the short-time limit, as we considered in Eqs. (39) and (40), Eqs. (41) and (42) become

$$\chi_{\alpha\beta}^S(t) \approx -k_B T \mu \left[\text{Sgn}(t) - \mu \frac{(k^e)^2 - (k^o)^2}{k^e} t \right] \delta_{\alpha\beta}, \quad (44)$$

$$\chi_{\alpha\beta}^A(t) \approx \bar{\chi}_{\alpha\beta} + 2k_B T \mu^2 k^o |t| \epsilon_{\alpha\beta}. \quad (45)$$

In Fig. 3, we plot $\chi_{11}(t)$ (black solid line) and $\chi_{12}(t)$ (red dashed line) as a function of dimensionless time $\mu k^e t$. The other parameter is $k^o/k^e = 1$ in Fig. 3(a) and $k^o/k^e = 3$ in Fig. 3(b). In both cases, $\chi_{11}(t)$ is an odd function, while $\chi_{12}(t)$ is an even function. In Fig. 3(b), we also observe an oscillatory behavior.

V. ODD ELASTICITY OF AN ENZYME SYSTEM WITH CHEMICAL REACTION

In this section, we discuss the application of our results to structural fluctuations in a model enzyme system introduced in Refs. 25, and 27–29 and also shown in Fig. 1(b). An enzyme changes its shape during a catalytic chemical reaction and receives chemical energy from a substrate molecule. This process is quite complicated and many degrees of freedom such as the positions of all the atoms are involved. To tackle such a problem, we first coarse grain the system to obtain the dynamical equations with minimum degrees of freedom such as in Eq. (29). Then, we assume that the energy injection from the substrate molecule is effectively described by the odd part of the elastic tensor. As an example of the application of our

analytical results, we use the time-correlation functions to obtain the effective odd elasticity of the enzyme model proposed by us.²⁵

In our model, we consider the dynamics of the extent of catalytic reaction $\theta(t)$ and the structure of an enzyme characterized by $s_1(t)$ and $s_2(t)$. The free energy describing a chemical reaction is given by $g_r(\theta) = -h \cos \theta - \nu \theta$, where h is the energy barrier in the chemical reaction and ν is the chemical potential difference. We also introduce the following mechano-chemical coupling energy:

$$g_c(\theta, \{s_i\}) = \frac{c}{2} ([s_1 - d \sin \theta]^2 + [s_2 - d \sin(\theta + p)]^2), \quad (46)$$

where c is the coupling strength, d is the amplitude of the structure change, and p is the phase difference relative to the reaction phase. The total free energy is given by $g_t(\theta, \{s_i\}) = g_r(\theta) + g_c(\theta, \{s_i\})$. The Onsager's phenomenological equations

$$\dot{\theta} = -\mu_\theta \partial_\theta g_t + \sqrt{2\mu_\theta} \xi, \quad (47)$$

$$\dot{s}_i = -\mu_s \partial_{s_i} g_t + \sqrt{2\mu_s} \xi_i \quad (48)$$

determine the time evolution of each variable. Here, ξ and ξ_i represent thermal fluctuations that satisfy Eq. (20). A more detailed explanation of our enzyme model is provided in Ref. 25.

We have performed numerical simulations of the above Langevin dynamics and calculated the structural time-correlation functions. We find that the time-reversal symmetry in Eq. (15) is broken in the correlation function $\phi_{12}(t) \sim \langle (s_1(t) - \langle s_1 \rangle)(s_2(0) - \langle s_2 \rangle) \rangle$ when $\nu \neq 0$. Comparing the short-time behavior of the simulation result with Eq. (C6), which is a generalized expression of Eq. (40), we have estimated the effective odd elasticity of the enzyme system. In Fig. 4, we plot the odd elasticity k^o as a

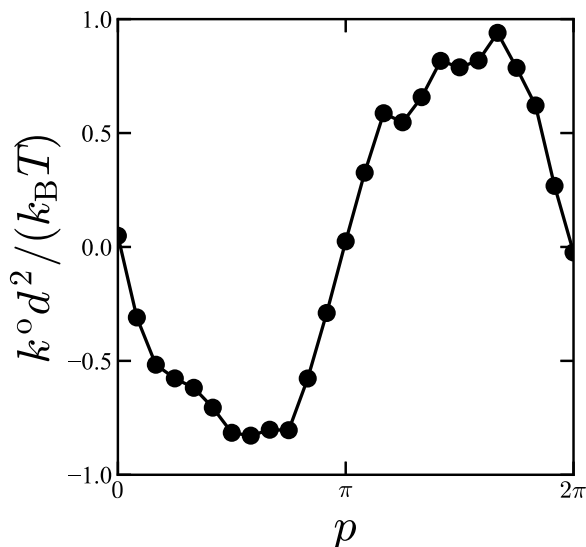


FIG. 4. Plot of the odd elasticity k^o estimated from the numerical simulations of the enzyme model in Ref. 25 as a function of the phase difference p . The chosen parameters are $\nu/(k_B T) = 16$, $h/(k_B T) = 20$, $cd^2/(k_B T) = 20$, and $\mu_s/(\mu_\theta d^2) = 1$.

function of p and find a periodic-dependence that can be approximately described by $k^o \sim -\sin p$. In our model, the phase difference p between the structural variables s_1 and s_2 is introduced to account for the nonreciprocal deformation of an enzyme molecule. Such a non-reciprocity can be quantitatively characterized by the area enclosed by a trajectory in a space spanned by s_1 and s_2 .²⁵ The relation $k^o \sim -\sin p$ indicates that the effective odd elasticity of an enzyme can be obtained only by measuring the structural dynamics without assuming any detailed dynamics of the internal variable such as θ .

A more detailed analysis of the simulation results will be presented in a separate publication. We emphasize here that such an analysis suggests a new possibility to understand nonequilibrium dynamics of active matter.

VI. SUMMARY AND DISCUSSION

In this paper, we have investigated the statistical properties of fluctuations in active systems that are governed by nonsymmetric responses. We first summarized the symmetry properties of the time-correlation matrices due to time-translational and time-reversal invariances. The antisymmetric parts of the time-correlation functions can exist in nonequilibrium situations. We investigated an underdamped Langevin system with a nonsymmetric resistance tensor and obtained the time-correlation matrices. We showed that time-reversal symmetry is violated in the presence of the odd part of the resistance tensor. For a system with two degrees of freedom, we calculated the analytical expressions of the time-correlation functions.

Next, we discussed an overdamped Langevin system with a nonsymmetric elastic tensor and obtained the corresponding time-correlation functions. We also showed that time-reversal symmetry of the correlation functions is violated in the presence of odd elasticity. The initial slope of the time-correlation functions represents the transport coefficient and the odd elasticity for the symmetric and the asymmetric parts, respectively. In particular, the position-velocity correlation function typically reflects the broken time-reversal symmetry and is proportional to the odd elasticity.

Let us give some numerical estimates of the physical quantities used in the present work. We consider the case when the concept of odd elasticity is applied to the structural changes of enzymes and motor proteins. The domain size of a protein is $a \approx 10^{-8}$ m and the viscosity of water is $\eta \approx 10^{-3}$ Pa s. Hence, the transport coefficient becomes $\mu = 1/(6\pi\eta a) \approx 5 \times 10^9$ m²/(J s). According to the experiments on a kinesin molecule,³⁰ the even elasticity can be roughly estimated as $k^e \approx 1 \times 10^{-4}$ J/m². Then, the relaxation rate can be roughly estimated as $\mu k^e \approx 5 \times 10^5$ s⁻¹. Next, we estimate the odd elastic constant from the activity of motor proteins. The active force due to kinesin is estimated to be $f \approx 10^{-11}$ N.^{30,31} By estimating the rough displacement to be $d \approx 10^{-8}$ m, the odd elastic constant can be estimated as $k^o \sim f/d \approx 10^{-3}$ J/m². Then, the ratio between the odd and even elastic constants can be typically $k^o/k^e \approx 10$.

In Secs. III and IV, we introduced the symmetric tensors $B_{\alpha\beta}$ and $D_{\alpha\beta}$ representing the noise strength. For general active situations, the noise originates not only from thermal fluctuations but also from nonequilibrium fluctuations. In active cases, $B_{\alpha\beta}$ and $D_{\alpha\beta}$ do not need to obey the fluctuation dissipation theorem and they

are general positive definite symmetric tensors. Another generalization is to take into account the antisymmetric parts of $B_{\alpha\beta}$ and $D_{\alpha\beta}$. In this situation, however, the time-reversal symmetry of noise [see Eq. (20)] can also be broken, and such a generalization is beyond the scope of the present work.

In Secs. III and IV, we have independently investigated the systems with nonsymmetric resistance tensor and nonsymmetric elastic tensor. When both of these properties exist simultaneously, the general Langevin equation can be written as

$$m\ddot{x}_\alpha = -\Lambda_{\alpha\beta}\dot{x}_\beta - K_{\alpha\beta}x_\beta + mF_{\alpha\beta}\xi_\beta(t), \quad (49)$$

where $\Lambda_{\alpha\beta}$ and $K_{\alpha\beta}$ are given by Eqs. (21) and (30), respectively. If the separation of time scales $m/\Lambda \ll 1/(\mu K)$ holds in a mesoscopic system, one can eliminate the inertia term and obtain the overdamped Langevin equation.⁷ In a thermal equilibrium system, the velocity-velocity correlation function $\psi_{\alpha\beta}(t)$ becomes a delta function in the limit of $m/\Lambda \rightarrow 0$. In nonequilibrium systems with odd properties, however, the time-reversal symmetry of $\psi_{\alpha\beta}(t)$ is violated as in Eq. (25). Hence, the time-correlation functions of the noise terms should involve not only the delta function but also its time derivative as discussed in Ref. 12.

Recently, the variational principle for active matter has been proposed.³² Within such an extended variational principle, one can obtain the dynamical equation in Eq. (29) by minimizing the Rayleighian $\mathcal{R} = (M^{-1})_{\alpha\beta}\dot{x}_\alpha\dot{x}_\beta/2 + \dot{A} + \dot{W}$, where $(M^{-1})_{\alpha\beta}$ is the inverse matrix of $M_{\alpha\beta}$. The first and second terms are the dissipation function and the time derivative of the free energy, respectively, while \dot{W} is the time derivative of the work generated by active forces. In our work, the even and odd elastic tensors can be included within the variational principle by choosing $\dot{A} = K_{\alpha\beta}^c\dot{x}_\alpha x_\beta$ and $\dot{W} = K_{\alpha\beta}^o\dot{x}_\alpha x_\beta$ to obtain Eq. (29). On the other hand, the odd resistance tensor cannot be described within the variational principle.

In the present work, we have discussed the linear Langevin equations with odd resistance tensor or odd elastic tensor. It should be noted, however, that we can also discuss nonlinear effects by considering a state-dependent resistance tensor $\Lambda_{\alpha\beta}^e(\{x_\alpha\})$ or a state-dependent mobility tensor $\mu_{\alpha\beta}(\{x_\alpha\})$ as well as nonlinear conservative forces. For example, nonlinearity appears in the dynamics of a deformable object in the presence of hydrodynamic interactions.^{26,33} For the future, the study of nonlinear dynamics in the presence of odd properties is necessary.

In this paper, we have discussed Langevin systems with either odd resistance tensor or odd elastic tensor. Owing to the mathematical analogy, the obtained results for the odd Langevin systems have various applications. Examples of Langevin systems with odd resistance tensor are the Brownian particle in a chiral active fluid (see Sec. III) and the Brownian particle under the Lorentz force.³⁴ On the other hand, odd elastic tensors can exist in situations such as those involving the structural dynamics of a catalytic enzyme (see Sec. V), Brownian particles in shear flow,³⁵ and stochastic behavior of the climate system.³⁶

To further strengthen our idea of effective odd elasticity, the following approaches will be useful. If we can generalize the projection operator formulation to nonequilibrium systems with chemical reactions, the coarse-grained Langevin equation such as

Eq. (29) can be obtained from the Hamilton dynamics by including all degrees of freedom.^{37–39} On the other hand, numerical simulations based on multiparticle collision dynamics can also be used to investigate the time-correlation functions of the enzyme structure.^{40,41}

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Kento Yasuda: Formal analysis (lead); Investigation (lead); Writing – original draft (lead); Writing – review & editing (lead). **Kenta Ishimoto:** Methodology (equal). **Akira Kobayashi:** Data curation (equal). **Li-Shing Lin:** Investigation (supporting). **Isamu Sou:** Investigation (supporting). **Yuto Hosaka:** Formal analysis (equal); Investigation (equal). **Shigeyuki Komura:** Conceptualization (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: GENERAL ANALYSIS OF THE LINEAR LANGEVIN EQUATION

In this appendix, we provide a general method to solve the linear Langevin equation with N degrees of freedom given by^{20,21}

$$\dot{X}_\alpha = A_{\alpha\beta}X_\beta + H_{\alpha\beta}\xi_\beta(t), \quad (A1)$$

where X_α represents any set of variables such as x_α or v_α , and $A_{\alpha\beta}$ is the $N \times N$ tensor characterizing the decay rate of the linear system. For the stability of the system, the real part of all eigenvalues of $A_{\alpha\beta}$ must be negative. The amplitude of the noise is given by a $N \times N$ tensor $H_{\alpha\beta}$ whose diagonal and off-diagonal components correspond to autocorrelations and cross-correlations of the noise,

respectively. Moreover, $\xi_\beta(t)$ is Gaussian white noise that satisfies Eq. (20). The above Langevin equation can be formally solved as

$$X_\alpha(t) = (e^{At})_{\alpha\beta} \int_{-\infty}^t ds (e^{-As})_{\beta\gamma} H_{\gamma\delta} \xi_\delta(s), \quad (\text{A2})$$

where we have used the matrix exponential $(e^{At})_{\alpha\beta}$. We can immediately confirm $\langle X_\alpha(t) \rangle = 0$.

Using the above solution, we can calculate the time-correlation functions $\Phi_{\alpha\beta}(t) = \langle X_\alpha(t) X_\beta(0) \rangle$ as

$$\Phi_{\alpha\beta}(t) = \begin{cases} (e^{At})_{\alpha\gamma} \bar{\Phi}_{\gamma\beta} & (t \geq 0), \\ \bar{\Phi}_{\alpha\gamma} (e^{-At})_{\beta\gamma} & (t < 0), \end{cases} \quad (\text{A3})$$

$$\bar{\Phi}_{\alpha\beta} = 2 \int_0^\infty ds (e^{As})_{\alpha\gamma} C_{\gamma\delta} (e^{As})_{\beta\delta}, \quad (\text{A4})$$

where $\bar{\Phi}_{\alpha\beta} = \Phi_{\alpha\beta}(0)$ are the equal-time-correlation functions and $C_{\alpha\beta} = H_{\alpha\gamma} H_{\beta\gamma} / 2$. Note that $\bar{\Phi}_{\alpha\beta}$ obeys the following Lyapunov equation:

$$A_{\alpha\gamma} \bar{\Phi}_{\gamma\beta} + A_{\beta\gamma} \bar{\Phi}_{\gamma\alpha} + 2C_{\alpha\beta} = 0. \quad (\text{A5})$$

The integral in Eq. (A4) can be calculated when $A_{\alpha\beta}$ is diagonalized by a matrix $P_{\alpha\beta}$. Then, $\bar{\Phi}_{\alpha\beta}$ is obtained by using the eigenvalues ζ_i of the matrix $A_{\alpha\beta}$ as

$$\bar{\Phi}_{\alpha\beta} = -2P_{\alpha\gamma} P_{\beta\delta} O_{\gamma\delta}, \quad (\text{A6})$$

$$O_{\gamma\delta} = (P^{-1})_{\gamma l} C_{lm} (P^{-1})_{\delta m} \circ \frac{1}{\zeta_\gamma + \zeta_\delta}, \quad (\text{A7})$$

where \circ stands for the Hadamard product (element-wise product). We note that $A_{\alpha\beta}$ cannot always be diagonalized when $A_{\alpha\beta}$ is a nonsymmetric matrix.

The time-correlation functions can be decomposed into the symmetric and antisymmetric parts as

$$\Phi_{\alpha\beta}(t) = \Phi_{\alpha\beta}^S(t) + \Phi_{\alpha\beta}^A(t), \quad (\text{A8})$$

$$\Phi_{\alpha\beta}^S(t) = \frac{1}{2} \left[(e^{A|t|})_{\alpha\gamma} \bar{\Phi}_{\gamma\beta} + (e^{A|t|})_{\beta\gamma} \bar{\Phi}_{\gamma\alpha} \right], \quad (\text{A9})$$

$$\Phi_{\alpha\beta}^A(t) = \frac{1}{2} \text{Sgn}(t) \left[(e^{A|t|})_{\alpha\gamma} \bar{\Phi}_{\gamma\beta} - (e^{A|t|})_{\beta\gamma} \bar{\Phi}_{\gamma\alpha} \right], \quad (\text{A10})$$

for both negative and positive t . We confirm here $\Phi_{\alpha\beta}^S(t) = \Phi_{\alpha\beta}^S(-t)$ and $\Phi_{\alpha\beta}^A(t) = -\Phi_{\alpha\beta}^A(-t)$.⁶ Moreover, we have $\bar{\Phi}_{\alpha\beta} = \bar{\Phi}_{\beta\alpha}$. When $\bar{\Phi}_{\alpha\beta} \propto \delta_{\alpha\beta}$ and $A_{\alpha\beta} \neq A_{\beta\alpha}$, we can easily confirm $\Phi_{\alpha\beta}^A(t) \neq 0$, which indicates the time-reversal symmetry breaking. The short-time asymptotic expressions of Eqs. (A9) and (A10) become

$$\Phi_{\alpha\beta}^S(t) \approx \bar{\Phi}_{\alpha\beta} + \frac{1}{2} [A_{\alpha\gamma} \bar{\Phi}_{\gamma\beta} + A_{\beta\gamma} \bar{\Phi}_{\gamma\alpha}] |t| = \bar{\Phi}_{\alpha\beta} - C_{\alpha\beta} |t|, \quad (\text{A11})$$

$$\Phi_{\alpha\beta}^A(t) \approx \frac{1}{2} [A_{\alpha\gamma} \bar{\Phi}_{\gamma\beta} - A_{\beta\gamma} \bar{\Phi}_{\gamma\alpha}] t. \quad (\text{A12})$$

For two degrees of freedom ($N = 2$), one can solve the Langevin equation analytically and $\bar{\Phi}_{\alpha\beta}$ is given by

$$\bar{\Phi}_{\alpha\beta} = -\frac{C_{\alpha\beta}}{\text{tr}[A]} - \frac{\det[A]}{\text{tr}[A]} (A^{-1})_{\alpha\gamma} C_{\gamma\delta} (A^{-1})_{\beta\delta}. \quad (\text{A13})$$

The time-dependence of the correlation functions can be obtained as

$$(e^{At})_{\alpha\beta} = e^{\gamma t} \left[\cosh(\omega t) \delta_{\alpha\beta} + \frac{A_{\alpha\beta} - \det[A] (A^{-1})_{\alpha\beta}}{2\omega} \sinh(\omega t) \right], \quad (\text{A14})$$

where we have introduced the relaxation rate $\gamma = \text{tr}[A]/2$ and the frequency $\omega = \sqrt{\gamma^2 - \det[A]}$.

APPENDIX B: CORRELATION FUNCTIONS IN UNDERDAMPED LANGEVIN SYSTEMS

In this appendix, we give the general expressions of the time-correlation functions for an underdamped system when $N = 2$. Here, $\Lambda_{\alpha\beta}^e$ is a symmetric and positive definite 2×2 matrix, while the odd part of the resistance tensor is given by $\Lambda_{\alpha\beta}^o = \lambda^o \epsilon_{\alpha\beta}$. Comparing Eqs. (19) and (A1), we obtain $A_{\alpha\beta} = -\Lambda_{\alpha\beta}/m$. From Eqs. (A9) and (A10), we then have

$$\psi_{\alpha\beta}^S(t) = \frac{k_B T}{2m} \left[(e^{-\Lambda|t|/m})_{\alpha\beta} + (e^{-\Lambda|t|/m})_{\beta\alpha} \right], \quad (\text{B1})$$

$$\psi_{\alpha\beta}^A(t) = \text{Sgn}(t) \frac{k_B T}{2m} \left[(e^{-\Lambda|t|/m})_{\alpha\beta} - (e^{-\Lambda|t|/m})_{\beta\alpha} \right], \quad (\text{B2})$$

where we have used the equipartition theorem in Eq. (22).

Furthermore, using Eq. (A14) and the relation $(\Lambda^{-1})_{\alpha\beta} = [\det[\Lambda^e] ((\Lambda^e)^{-1})_{\alpha\beta} - \lambda^o \epsilon_{\alpha\beta}] / \det[\Lambda]$ for a 2×2 matrix, we obtain

$$\psi_{\alpha\beta}^S(t) = \frac{k_B T}{m} e^{-\gamma|t|} \left[\cosh(\omega t) \delta_{\alpha\beta} - \frac{\sinh(\omega|t|)}{2\omega m} [\Lambda_{\alpha\beta}^e - \det[\Lambda^e] ((\Lambda^e)^{-1})_{\alpha\beta}] \right], \quad (\text{B3})$$

$$\psi_{\alpha\beta}^A(t) = -\frac{k_B T}{m^2 \omega} \lambda^o e^{-\gamma|t|} \sinh(\omega t) \epsilon_{\alpha\beta}. \quad (\text{B4})$$

In the above, we have defined the relaxation rate $\gamma = \text{tr}[\Lambda^e]/(2m)$ and the frequency $\omega = \sqrt{\gamma^2 m^2 - \det[\Lambda^e] - (\lambda^o)^2}/m$.

The exceptional point is given by the condition $\omega = 0$ and we obtain $\lambda_{\text{ep}}^o = \pm \sqrt{\gamma^2 m^2 - \det[\Lambda^e]}$.⁴² The frequency ω is a real number when $(\lambda^o)^2 < (\lambda_{\text{ep}}^o)^2$, while it is an imaginary number when $(\lambda^o)^2 > (\lambda_{\text{ep}}^o)^2$ for which the time-correlation function can oscillate. To see an oscillating behavior, however, we further need a condition $\omega^2 < -\gamma^2$, as shown in Fig. 2(b).

APPENDIX C: CORRELATION FUNCTIONS IN OVERDAMPED LANGEVIN SYSTEMS

In this appendix, we give the general expressions of the time-correlation functions for an overdamped system when $N = 2$. Here,

both $M_{\alpha\beta}$ and $K_{\alpha\beta}^e$ are symmetric and positive definite 2×2 matrices, while the odd part of the elastic tensor is given by $K_{\alpha\beta}^o = k^o \epsilon_{\alpha\beta}$. Comparing Eqs. (29) and (A1), we obtain $A_{\alpha\beta} = -M_{\alpha\gamma} K_{\gamma\beta}$ and $C_{\alpha\beta} = D_{\alpha\beta} = k_B T M_{\alpha\beta}$. From Eq. (A13), $\bar{\phi}_{\alpha\beta}$ becomes

$$\bar{\phi}_{\alpha\beta} = \frac{k_B T M_{\alpha\beta}}{\text{tr}[MK^e]} + k_B T \frac{\det[M][\det[K^e] + (k^o)^2]}{\text{tr}[MK^e]} \times (K^{-1})_{\alpha\gamma} (M^{-1})_{\gamma\delta} (K^{-1})_{\delta\beta}. \quad (\text{C1})$$

Furthermore, using the relation $(K^{-1})_{\alpha\beta} = [\det[K^e]((K^e)^{-1})_{\alpha\beta} - k^o \epsilon_{\alpha\beta}]/[\det[K^e](1 + v^2)]$ with $v^2 = (k^o)^2/\det[K^e]$, we obtain the following expression:

$$\bar{\phi}_{\alpha\beta} = \frac{k_B T}{1 + v^2} \left[((K^e)^{-1})_{\alpha\beta} + \frac{2v^2}{\text{tr}[MK^e]} M_{\alpha\beta} - \frac{k^o \det[M]}{\text{tr}[MK^e]} \times [\epsilon_{\alpha\gamma} (M^{-1})_{\gamma\delta} ((K^e)^{-1})_{\delta\beta} + \epsilon_{\beta\gamma} (M^{-1})_{\gamma\delta} ((K^e)^{-1})_{\delta\alpha}] \right]. \quad (\text{C2})$$

In the above, we have used the identities $M_{\alpha\beta} + \det[M](M^{-1})_{\alpha\beta} = \text{tr}[M]\delta_{\alpha\beta}$ and $\det[M]\epsilon_{\alpha\gamma}(M^{-1})_{\gamma\delta}\epsilon_{\delta\beta} = M_{\beta\alpha}$ for a 2×2 matrix. When $k^o = 0$ and hence $v = 0$, the above expression reduces to $\bar{\phi}_{\alpha\beta} = k_B T ((K^e)^{-1})_{\alpha\beta}$ corresponding to the thermal equilibrium case.

The time-dependence of the correlation functions is calculated by using Eq. (A14),

$$(e^{-MK|t|})_{\alpha\beta} = e^{-\gamma|t|} \left[\cosh(\omega t) \delta_{\alpha\beta} - \frac{M_{\alpha\delta} K_{\delta\beta} - \det[M] \det[K] (K^{-1})_{\alpha\delta} (M^{-1})_{\delta\beta}}{2\omega} \times \sinh(\omega|t|) \right], \quad (\text{C3})$$

where we have introduced the relaxation rate $\gamma = \text{tr}[MK^e]/2$ and the frequency $\omega = \sqrt{\gamma^2 - \det[M] \det[K^e](1 + v^2)}$.

The symmetric and antisymmetric parts of the correlation matrix can be obtained from Eqs. (A9) and (A10), respectively, as

$$\phi_{\alpha\beta}^S(t) = e^{-\gamma|t|} \cosh(\omega t) \bar{\phi}_{\alpha\beta} - \frac{k_B T e^{-\gamma|t|} \sinh(\omega|t|)}{2\omega(1 + v^2) \text{tr}[MK^e]} \times [M_{\alpha\gamma} K_{\gamma\delta}^e M_{\delta\beta} - (\det[M])^2 (\det[K^e])^2 ((K^e)^{-1})_{\alpha\gamma} \times (M^{-1})_{\gamma\delta} ((K^e)^{-1})_{\delta\ell} (M^{-1})_{\ell m} ((K^e)^{-1})_{m\beta} + k^o (\det[M])^2 \det[K^e] [\epsilon_{\alpha\gamma} (M^{-1})_{\gamma\delta} ((K^e)^{-1})_{\delta\ell} \times (M^{-1})_{\ell m} ((K^e)^{-1})_{m\beta} + \epsilon_{\beta\gamma} (M^{-1})_{\gamma\delta} ((K^e)^{-1})_{\delta\ell} \times (M^{-1})_{\ell m} ((K^e)^{-1})_{m\alpha}]], \quad (\text{C4})$$

$$\phi_{\alpha\beta}^A(t) = -\frac{2k_B T e^{-\gamma|t|} k^o \sinh(\omega t)}{\omega \text{tr}[MK^e]} \det[M] \epsilon_{\alpha\beta}. \quad (\text{C5})$$

In the short-time limit, Eqs. (C4) and (C5) asymptotically become

$$\phi_{\alpha\beta}^S(t) \approx \bar{\phi}_{\alpha\beta} - k_B T M_{\alpha\beta} |t|, \quad \phi_{\alpha\beta}^A(t) \approx -\frac{2k_B T k^o t}{\text{tr}[MK^e]} \det[M] \epsilon_{\alpha\beta}. \quad (\text{C6})$$

Similar to Eqs. (39) and (40), the slopes of the symmetric and anti-symmetric parts are proportional to the mobility tensor $M_{\alpha\beta}$ and the odd elasticity k^o , respectively, although the results are more general.

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