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Short Notes

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Odd Microswimmer

Kento Yasuda*, Yuto Hosaka,
Isamu Sou, and Shigeyuki Komura

*Department of Chemistry, Graduate School of Science,
Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan*

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We propose a model for a thermally driven microswimmer in which three spheres are connected by two springs with odd elasticity. We demonstrate that the presence of odd elasticity leads to the directional locomotion of the stochastic microswimmer.

Although micromachines such as proteins and enzymes experience the influence of strong thermal fluctuations, they often exhibit directional locomotion under nonequilibrium conditions.[1] To elucidate this type of phenomena, we previously proposed a thermally driven elastic microswimmer consisting of three spheres.[2] In this model, the three spheres were assumed to be in equilibrium with independent heat baths characterized by different temperatures.

Recently, Scheibner et al. introduced the concept of "odd elasticity", which can arise from active and nonreciprocal interactions.[3] Importantly, the odd part of the elastic constant tensor quantifies the amount of work extracted along quasistatic deformation cycles. In this paper, we propose a novel type of thermally driven microswimmer in which the three spheres are connected with springs having not only even elasticity,[4] but also odd elasticity.[3] We explicitly demonstrate that the proposed stochastic "odd microswimmer" can exhibit a directional locomotion as a result of odd elasticity. Additionally, we provide a simple physical interpretation of the average velocity within the nonequilibrium statistical physics.

Consider a three-sphere microswimmer in which the positions of the three spheres of radius a are given by x_i ($i = 1, 2, 3$) in a one-dimensional coordinate system (see Fig. 1).[5] These three spheres are connected by two springs that exhibit both even and odd elasticity. We denote the two spring extensions as $u_A = x_2 - x_1 - \ell$ and $u_B = x_3 - x_2 - \ell$, where ℓ is the natural length. Then, the forces F_A and F_B conjugate to u_A and u_B, respectively, are given by $F_\alpha = -K_{\alpha\beta}u_\beta$ ($\alpha, \beta = \mathrm{A}, \mathrm{B}$). For an odd spring, the elastic constant $K_{\alpha\beta}$ is given by[3]

$$K_{\alpha\beta} = K^{\mathrm{e}}\delta_{\alpha\beta} + K^{\mathrm{o}}\epsilon_{\alpha\beta}, \tag{1}$$

where K^{e} and K^{o} are the even and odd elastic constants, respectively, in the 2D configuration space spanned by u_A and u_B (unlike the real 2D space in Ref. 3), $\delta_{\alpha\beta}$ is the Kronecker delta, and $\epsilon_{\alpha\beta}$ is the 2D Levi-Civita tensor with $\epsilon_{AA} = \epsilon_{BB} = 0$ and $\epsilon_{AB} = -\epsilon_{BA} = 1$. The presence of odd elasticity K^{o} in Eq. (1) reflects the nonreciprocal interaction between the two springs such that u_A and u_B influence each other in a different manner.[6] The forces f_i acting on each sphere are given by $f_1 = -F_A$, $f_2 = F_A - F_B$, and $f_3 = F_B$. These forces satisfy the force-free condition, i.e., $f_1 + f_2 + f_3 = 0$.

The odd microswimmer described above is immersed in a fluid with a shear viscosity of η and temperature T. Then the equations of motion for each sphere are given by[2,4,5]

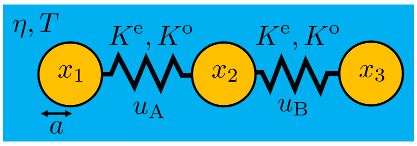

Fig. 1. (Color online) Odd microswimmer in a fluid with a viscosity η and temperature T. Three spheres of radius a are connected by two springs with a natural length ℓ. Each spring has both even elastic constant K^{e} and odd elastic constant K^{o}. The positions of the spheres are denoted as x_i ($i = 1, 2, 3$), and the spring extensions with respect to ℓ are denoted as u_A and u_B.

$$\dot{x}_i = M_{ij}f_j + \xi_i, \tag{2}$$

where $\dot{x}_i = dx_i/dt$ and M_{ij} are the hydrodynamic mobility coefficients[5]

$$M_{ij} = \begin{cases} 1/(6\pi\eta a) & (i = j) \\ 1/(4\pi\eta|x_i - x_j|) & (i \neq j) \end{cases}. \tag{3}$$

In Eq. (2), the Gaussian white-noise sources ξ_i have zero mean $\langle\xi_i(t)\rangle = 0$, and their correlations satisfy the following fluctuation–dissipation theorem:

$$\langle\xi_i(t)\xi_j(t')\rangle = 2k_BTM_{ij}\delta(t - t'), \tag{4}$$

where k_B is the Boltzmann constant.

It is convenient to introduce the characteristic time scale $\tau = 6\pi\eta a/K^{\mathrm{e}}$ and the ratio between the two spring constants $\lambda = K^{\mathrm{o}}/K^{\mathrm{e}}$. In the following analysis, we assume $u_A, u_B \ll \ell$ and $a \ll \ell$, and focus solely on the leading-order contribution. The total velocity of the microswimmer is given by $V = (\dot{x}_1 + \dot{x}_2 + \dot{x}_3)/3$. After taking the statistical average and using Eqs. (1)–(3), we obtain[2]

$$\langle V\rangle = \frac{a}{8\ell^2\tau}[\langle u_B^2\rangle - \langle u_A^2\rangle + \lambda(3\langle u_B^2\rangle + 3\langle u_A^2\rangle - 2\langle u_Au_B\rangle)] + \mathcal{O}[(a/\ell)^2, (u/\ell)^3], \tag{5}$$

where we use $\langle u_A\rangle = \langle u_B\rangle = 0$.

The equal-time correlation functions appearing in Eq. (5) can be obtained from the reduced Langevin equations for $\dot{u}_A = \dot{x}_2 - \dot{x}_1$ and $\dot{u}_B = \dot{x}_3 - \dot{x}_2$ as

$$\dot{u}_\alpha = \Gamma_{\alpha\beta}u_\beta + \Xi_\alpha + \mathcal{O}[a/\ell], \tag{6}$$

where $\Gamma_{\alpha\beta}$ and Ξ_α are

$$\mathbf{\Gamma} = -\frac{1}{\tau}\begin{pmatrix} 2 + \lambda & -1 + 2\lambda \\ -1 - 2\lambda & 2 - \lambda \end{pmatrix}, \quad \mathbf{\Xi} = \begin{pmatrix} \xi_2 - \xi_1 \\ \xi_3 - \xi_2 \end{pmatrix}. \tag{7}$$

Notice that $\Gamma_{\alpha\beta}$ is nonreciprocal, i.e., $\Gamma_{AB} \neq \Gamma_{BA}$ when $\lambda \neq 0$. By solving Eq. (6) in the Fourier domain and using Eq. (4), we obtain the following equal-time correlation functions:[2]

$$\langle u_A^2\rangle = \frac{k_BT}{K^{\mathrm{e}}}\left[1 - \frac{\lambda}{2(1 + \lambda^2)}\right] + \mathcal{O}[a/\ell], \tag{8}$$

$$\langle u_B^2\rangle = \frac{k_BT}{K^{\mathrm{e}}}\left[1 + \frac{\lambda}{2(1 + \lambda^2)}\right] + \mathcal{O}[a/\ell], \tag{9}$$

$$\langle u_Au_B\rangle = -\frac{k_BT}{K^{\mathrm{e}}}\frac{\lambda^2}{2(1 + \lambda^2)} + \mathcal{O}[a/\ell]. \tag{10}$$

Here, we neglect the cross-correlations $\langle\xi_i\xi_j\rangle$ with $i \neq j$ because they only contribute to higher orders in a/ℓ. When $\lambda = 0$, the above expressions reduce to $\langle u_A^2\rangle = \langle u_B^2\rangle = k_BT/K^{\mathrm{e}}$ and $\langle u_Au_B\rangle = 0$, reproducing the thermal equilibrium situation. We have $\langle u_A^2\rangle < \langle u_B^2\rangle$ when $\lambda > 0$, because the effective elastic constant of spring A is greater than that of spring B.

By substituting Eqs. (8)–(10) into Eq. (5), we obtain the average velocity as

$$\langle V \rangle = \frac{7ak_{\mathrm{B}}T\lambda}{8\ell^2 K^{\mathrm{e}}\tau} + \mathcal{O}[(a/\ell)^2, (u/\ell)^3]. \quad (11)$$

Here, $\langle V \rangle$ is proportional to the odd elastic constant K^{o} that can take either positive or negative value. Because $\langle V \rangle$ is also proportional to $k_{\mathrm{B}}T$, thermal fluctuations are responsible for the locomotion of the odd microswimmer. Therefore, our model provides a novel type of Brownian ratchet.

Next, we discuss the nonequilibrium statistical properties of the odd microswimmer.[7,8] For the time-dependent probability distribution function $p(u_{\mathrm{A}}, u_{\mathrm{B}}, t)$, the Fokker–Planck equation corresponding to Eq. (6) can be written as $\dot{p} = -\partial_\alpha j_\alpha$, where $\partial_\alpha = \partial/(\partial u_\alpha)$ and j_α is the probability flux given by[7]

$$j_\alpha = \Gamma_{\alpha\beta} u_\beta p - D_{\alpha\beta}\partial_\beta p. \quad (12)$$

Here, $D_{\alpha\beta}$ is the diffusion matrix

$$\mathbf{D} = \frac{k_{\mathrm{B}}T}{6\pi\eta a}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (13)$$

which satisfies the relationship $\langle \Xi_\alpha(t)\Xi_\beta(t')\rangle = 2D_{\alpha\beta}\delta(t-t')$ according to Eq. (4).

Owing to the reproductive property of Gaussian distributions, the steady-state probability distribution function that satisfies $\dot{p} = 0$ is given by a Gaussian function[7]

$$p(u_{\mathrm{A}}, u_{\mathrm{B}}) = \frac{1}{2\pi\sqrt{\det \mathbf{C}}}\exp\left[-\frac{1}{2}(C^{-1})_{\alpha\beta}u_\alpha u_\beta\right]. \quad (14)$$

Here, $C_{\alpha\beta} = \langle u_\alpha u_\beta \rangle$ is the covariance matrix obtained from Eqs. (8)–(10) as

$$\mathbf{C} = \frac{k_{\mathrm{B}}T}{K^{\mathrm{e}}}\frac{1}{1+\lambda^2}\begin{pmatrix} 1-\lambda/2+\lambda^2 & -\lambda^2/2 \\ -\lambda^2/2 & 1+\lambda/2+\lambda^2 \end{pmatrix}, \quad (15)$$

and $(C^{-1})_{\alpha\beta}$ is the inverse matrix of $C_{\alpha\beta}$. For our purposes, we explicitly show that

$$\det \mathbf{C} = \left(\frac{k_{\mathrm{B}}T}{K^{\mathrm{e}}}\right)^2 \frac{4+7\lambda^2+3\lambda^4}{4(1+\lambda^2)^2}. \quad (16)$$

In Fig. 2, we plot the steady-state probability distribution function in Eq. (14) and corresponding probability flux in Eq. (12) when $\lambda = 1$. The probability distribution function is distorted by the negative correlation ($C_{\mathrm{AB}} = C_{\mathrm{BA}} \sim -\lambda^2/2$) between u_{A} and u_{B}. One can see a counter-clockwise loop of the probability flux. Such a probability flux becomes clockwise for $\lambda < 0$ and vanishes when $\lambda = 0$. The existence of a probability flux loop indicates that the detailed balance is broken in the nonequilibrium steady state.

The steady-state probability flux can be conveniently expressed in terms of a frequency matrix $\Omega_{\alpha\beta}$ as $j_\alpha = \Omega_{\alpha\beta}u_\beta p$.[7] For the proposed odd microswimmer, the frequency matrix is given by

$$\boldsymbol{\Omega} = \frac{3\lambda}{\tau(4+3\lambda^2)}\begin{pmatrix} -\lambda^2 & -2+\lambda-2\lambda^2 \\ 2+\lambda+2\lambda^2 & \lambda^2 \end{pmatrix}, \quad (17)$$

which is traceless. Then, the two eigenvalues of $\Omega_{\alpha\beta}$ are given by

$$\gamma = \pm \mathrm{i}\frac{3\lambda}{\tau(4+3\lambda^2)}\sqrt{4+7\lambda^2+3\lambda^4}. \quad (18)$$

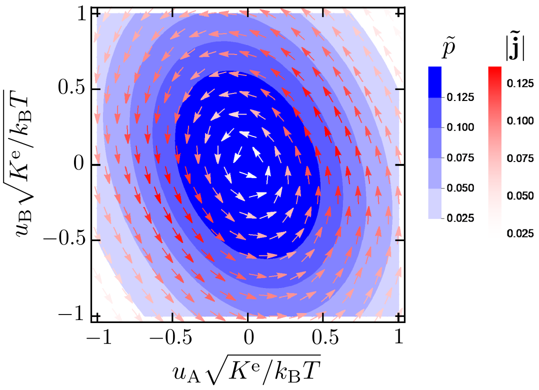

Fig. 2. (Color online) Steady-state scaled probability distribution function $\tilde{p} = pk_{\mathrm{B}}T/K^{\mathrm{e}}$ and steady-state scaled probability flux $\tilde{\mathbf{j}} = \mathbf{j}\tau\sqrt{k_{\mathrm{B}}T/K^{\mathrm{e}}}$ (arrows) in the configuration space spanned by u_{A} and u_{B} when $\lambda = K^{\mathrm{o}}/K^{\mathrm{e}} = 1$.

Because these eigenvalues are purely imaginary, the probability current in the configuration space is rotational. Comparing Eq. (11) with Eqs. (16) and (18), we obtain the following simple expression for the average velocity:

$$|\langle V \rangle| = \frac{7a}{12\ell^2}\sqrt{\det \mathbf{C}}\,|\gamma|. \quad (19)$$

Here, $7a/(12\ell^2)$ is the geometrical factor,[5] $\sqrt{\det \mathbf{C}} \sim k_{\mathrm{B}}T/K^{\mathrm{e}}$ is the explored area in the configuration space, and $|\gamma| \sim \tau^{-1}$ is the speed of the rotational probability flux.[7]

Finally, we consider the work that can be extracted when odd elasticity exsits.[3] For the stochastic odd microswimmer, the average power can be evaluated as $\langle \dot{W} \rangle = -K_{\alpha\beta}\langle \dot{u}_\alpha u_\beta \rangle$, where $W = \int du_\alpha F_\alpha$. From Eq. (6), we obtain $\langle \dot{u}_{\mathrm{A}} u_{\mathrm{B}} \rangle = -\langle \dot{u}_{\mathrm{B}} u_{\mathrm{A}} \rangle = -3k_{\mathrm{B}}T\lambda/(2K^{\mathrm{e}}\tau)$ and $\langle \dot{u}_{\mathrm{A}} u_{\mathrm{A}} \rangle = \langle \dot{u}_{\mathrm{B}} u_{\mathrm{B}} \rangle = 0$. By using these results, we can estimate the power of the odd microswimmer as $\langle \dot{W} \rangle = 3k_{\mathrm{B}}T\lambda^2/\tau$. We have confirmed that this power coincides with the average entropy production rate obtained by the expression $\langle \dot{\sigma} \rangle = -\mathrm{Tr}[\boldsymbol{\Gamma}(\boldsymbol{\Gamma}\mathbf{C}\mathbf{D}^{-1} + \mathbf{I})]$,[8] where $\mathbf{I}$ is the identity matrix. Therefore, all the extracted work due to odd elasticity is converted into the entropy production. It is also useful to note that the average velocity can be alternatively written as $\langle V \rangle = 7a/(12\ell^2)\langle \dot{u}_{\mathrm{B}} u_{\mathrm{A}} \rangle$.

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*Present address: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

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